University of Victoria

CSC 320 - Spring 2023

Foundations of Computer Science

Tutorial 04

Teaching Team

Learning Outcomes:

- Understand the Pumping Lemma.
- Prove that a language is not regular using the Pumping Lemma.

Interesting Article:

"Pumping Lemma for Quantum Automata" [1]

February 7th, 2023

Question 01

Prove that the following language is not regular using the pumping lemma.

$$L_1 = \{0^n 1^n 2^n \mid n \geq 0\}$$

We remember th general structure of th pumping lemma:

Let us observe th following...

$w \in L$, $|w| \geq p$, $w = xyz$, such that:

1. $|xy| \leq p$
2. $y \neq \varepsilon$
3. $xy^k z \in L$, $k \geq 0$.

Important to state!

Assume L_1 is regular. Then for ALL strings in L_1 with length greater than or equal to the pumping length, the pumping lemma holds

Declare your variables!

Pick: $w = 0^p 1^p 2^p$ where p is th pumping length.

$w \in L_1$ since there are th same amounts of 0's, 1's, and 2's and $|w| \geq p$.

We know this since "IF" we give p a value (purely as an example and not a part of th proof) of 3 then $0^3 1^3 2^3 = 000111222$ which statifies $0^n 1^n 2^n$, where $n \geq 0$ AND $3(3) = 9$ which is greater than or equal to p.

By th pumping lemma, $w = xyz$

$\underbrace{0 \ldots 0}_{p} \underbrace{1 \ldots 1}_{p} \underbrace{2 \ldots 2}_{p}$

① $|xy| \leq p$, so xy contains only 0's.

② $y \neq \varepsilon$, so $y = 0^i$ for some $i > 0$.

Variable declaration!

③ $xy^k z \in L_1$, $k \geq 0$

Case 1: $xy^0 z \notin L_1$, since $0^{p-i} 1^p 2^p \notin L_1$.

Case 2: $xy^2 z \notin L_1$, since $0^{p+i} 1^p 2^p \notin L_1$.

We obtain a contradiction!

Thus, we can conclude that L_1 is not regular. ∎

Don't forget th concluding sentence!

Question 02

Prove that the following language is not regular using the pumping lemma.

$$L_2 = \{w^r w \mid w \in \{0,1\}^*\}$$

Start with our intuition of what is and isn't in the language:

NOT : 01, 10, 110, 011,

IN : 0110, 1001, 11, 00, 101101,

Assume L_2 is regular. Then for ALL strings in L_2 with length $\geq$ pumping length, the pumping lemma holds.

← Logic that we base the whole proof on to prove it is NOT regular.

Pick $w = 0^p110^p$ where p is the pumping length.

so, $w \in L_2$ and $|w| \geq p$.

Since 001100, 00011000 are all in the language and $p+1+1+p = 2p+2 \geq p$.

By the pumping lemma, $w = xyz$.

1. $|xy| \leq p$, so xy contains only 0's.
2. $y \neq \epsilon$, so $y = 0^i$, for some $i > 0$.
3. $xy^kz \in L_2$, $k \geq 0$

$$\underbrace{0\ldots0}_{p}\,11\,\underbrace{0\ldots0}_{p}$$

↙ Remember to do all cases!

Case 1: $xy^0z \notin L_2$ since $0^{p-i}110^p \notin L_2$.

Case 2: $xy^2z \notin L_2$ since $0^{p+i}110^p \notin L_2$.

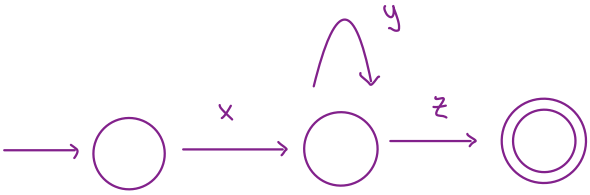
$$\underbrace{0\ldots0}_{p-i}\,11\,\underbrace{0\ldots0}_{p}$$

Thus, we no longer have $w^r w$, i.e., 0011000 which is not in L_2.

We obtain a contradiction!

Therefore, L_2 is not regular. ∎

Remember



Assume $w = xyz$, so we want to be able to "pump" y which still ends in an accept state for ALL w.

Note: Which other strings might be used in our proof?

For example, is 1^p001^p a good string? (yes ☺).

Reflection on Question 02

Why is the string $s = 0^P0^P$ not a good choice to devise a contradiction to prove L_2 is not regular?

Thoughts

Why did we chose $0^p 11 p^2$ and not something else? How did we devise such string?
What happens if we choose the wrong string?

Intuitively

We chose $0^p 11 0^p$ since we need 1's to "separate" the 0's so when we pump y in rule 3, we show that we cannot be of form $w^r w$ since the number of 0's before 1's will be different than the number of 0's after the 1's.

If we didn't have 1's to separate the 0's we could get in a situation like...

$$\underbrace{00}\,\underbrace{00}_{y}\,\underbrace{00} \;\Rightarrow\; y^0 \quad \underbrace{00}_{w^r}\,\underbrace{00}_{w}$$

Where we can still write the resulting string in the form $w^r w$.

Formally

Assume L_2 is regular. Then for all strings in L_2 with length $\geq$ pumping length, the pumping lemma holds.

Pick $w = 0^p p^p$ where p is the pumping length.

So, $w \in L_2$ and $|w| \geq p$. $p + p = 2p \geq p$

By the pumping lemma, $w = xyz$.

1. xy is all 0s in the first half of 0's $\quad \underbrace{0\ldots0}_{p}\,\underbrace{0\ldots0}_{p}$
2. $y = 0^i$, $i > 0$
3. $xy^k z \in L_2$, $k \geq 0$

We want to derive a contradiction, but we cannot guarantee that pumping y up or down results in a string not in L_2.

If i is even, then:

$i = 2j$, $j > 0$

Case 1 $\quad xy^0z = 0^{p-2j}0^p = 0^{2p-2j} = 0^{2(p-j)}$
$= 0^{p-j}0^{p-j}$,

which we know is in L_2. $\quad p = 3$, $j = 2$, which gives us $0^{3-2}0^{3-2} = 0^1 0^1$, which is of the form $w^r w$.

Case 2 $\quad xy^m z$, $m > 1 \Rightarrow xy^m z = 0^{p+m(2j)}0^p = 0^{2p+2mj}$
$= 0^{p+mj}0^{p+mj}$

which is also in L_2.

So, we cannot use $w = 0^p0^p$, because we do not reach a contradiction. ▲

Note: Convince yourself that 1^p1^p would give us the same result as above.

Resources

[1] R. Lu and H. Zheng, "Pumping lemma for quantum automata," English, *International journal of theoretical physics*, vol. 43, no. 5, pp. 1191–1217, 2004, ISSN: 0020-7748.