SENG 480D - Quantum Algorithms and Software Engineering

Theory and implementation of quantum algorithms including challenges and opportunities for quantum software developers and engineers using Qiskit, Q# SDK, Jupyter Notebooks, and Interactive Textbooks.

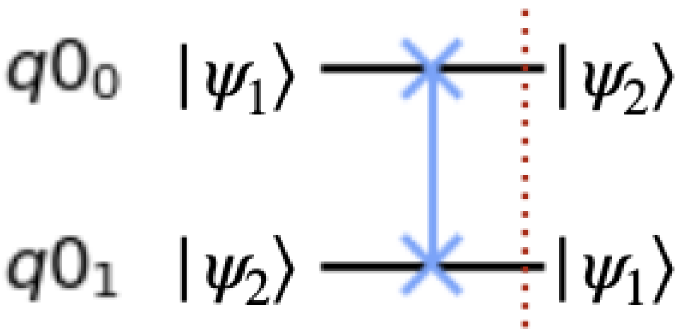
Lecture 05 Slide Deck:

- lecture-05-slide-deck-01.pdf

Slide Deck 01A

SWAP-Gate

$$\text{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Input $\lvert\phi\rangle$	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 11\rangle$
SWAP $\lvert\phi\rangle$	$\lvert 00\rangle$	$\lvert 10\rangle$	$\lvert 01\rangle$	$\lvert 11\rangle$

Let $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ and $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$ (i.e., $\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ respectively). Then...

$$SWAP\ |\phi_1\phi_2\rangle = SWAP((\alpha_1|0\rangle + \beta_1|1\rangle)\ \otimes\ (\alpha_2|0\rangle + \beta_2|1\rangle))$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \\ \beta 1 & \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ \beta_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_2\alpha_1 \\ \alpha_2\beta_1 \\ \beta_2\alpha_1 \\ \beta_2\beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$$

$$= ((\alpha_2|0\rangle + \beta_2|1\rangle)\ \otimes\ (\alpha_1|0\rangle + \beta_1|1\rangle))$$

The SWAP gate is hermitian.

Controlled Z-Gate

$$C_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

where, the Identity matrix is represented in purple and the Z-Gate matrix is represented in green.

Input	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
C_Z	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$-\|11\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ -\delta \end{bmatrix}$$

Thus,

$$C_Z(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) \\ = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle - \delta|11\rangle$$

The C_Z gate is hermitian.

Quantum Gates and their Controlled Relatives

Given a 1-qubit quantum gate $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ the controlled U-Gate C_U is the matrix...

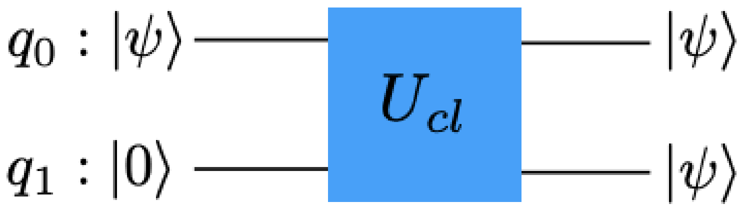
$$C_U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}$$

Slide Deck 01B

No Cloning Theorem

There is no unitary operator that can create a copy of any quantum state.

Copying: What must a cloning operator look like?



The quantum copy gate would function as follows:

$$U_{cl}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for any quantum state $|\psi\rangle$, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

Now, does such an operator exist such that this is true for any $|\psi\rangle$?

Left Hand Side

Consider **left hand side** of the circuit:

$$\begin{aligned} U_{cl}(|\psi\rangle \otimes |0\rangle) &= U_{cl}((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle) \\ &= (\alpha|0\rangle \otimes |0\rangle + \beta|1\rangle) \otimes |0\rangle) \\ &= \alpha U_{cl}(|0\rangle \otimes |0\rangle) + \beta U_{cl}(|1\rangle \otimes |0\rangle) \end{aligned}$$

Using the definition of U_{cl}

$$= \alpha|00\rangle + \beta|1\rangle$$

Right Hand Side

The **right hand side** of the circuit:

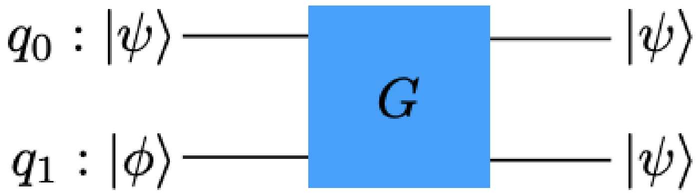
$$\begin{aligned} |\psi\rangle \otimes |\psi\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle \end{aligned}$$

Thus, a **contradiction** unless $\alpha = 0$ or $\beta = 0$.

No Cloning Theorem

There is no unitary operator that can create a copy of any quantum state.

Copying: How about a different initialization?



If G, $|\phi\rangle$ satisfy that:

$$G(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for any $|\psi\rangle$. Then...

$$|x_1\rangle = G(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

$$\text{and}$$

$$|x_2\rangle = G(|\psi'\rangle \otimes |\phi\rangle) = |\psi'\rangle \otimes |\psi'\rangle.$$

Left Hand Side

Consider the inner product...

$$\begin{aligned}
\langle x_1|x_2\rangle &= (G(|\psi\rangle \otimes |\phi\rangle))^\dagger (G(|\psi'\rangle \otimes |\phi\rangle)) \\
&= (|\psi\rangle \otimes |\phi\rangle)^\dagger G^\dagger G(|\psi'\rangle \otimes |\phi\rangle) \\
&= (|\psi\rangle \otimes |\phi\rangle)^\dagger (|\psi'\rangle \otimes |\phi\rangle) \\
&= (|\psi\rangle^\dagger \otimes |\phi\rangle^\dagger)(|\psi'\rangle \otimes |\phi\rangle) \\
&= (\langle\psi| \otimes \langle\phi|)(|\psi'\rangle \otimes |\phi\rangle) \\
&= \langle\psi|\psi'\rangle \langle\phi|\phi\rangle \\
&= \langle\psi|\psi'\rangle
\end{aligned}$$

where,

$$\langle\psi|\psi'\rangle = \langle\psi|\psi'\rangle^2 \iff \langle\psi|\psi'\rangle = 0 \quad \text{or} \quad \langle\psi|\psi'\rangle = 1$$

$$\langle\psi|\psi'\rangle = \langle\psi|\psi'\rangle^2 \iff |\psi\rangle \text{ and } |\psi'\rangle \text{ are orthogonal} \quad \text{or} \quad |\psi\rangle = |\psi'\rangle$$

Right Hand Side

Consider...

$$\begin{aligned}
(|\psi\rangle \otimes |\psi\rangle)^\dagger (|\psi'\rangle \otimes |\psi'\rangle) &= \\
(\langle\psi| \otimes \langle\psi|)(|\psi'\rangle \otimes |\psi'\rangle) &= \\
\langle\psi|\psi'\rangle \langle\psi|\psi'\rangle &=
\end{aligned}$$

No Cloning Theorem

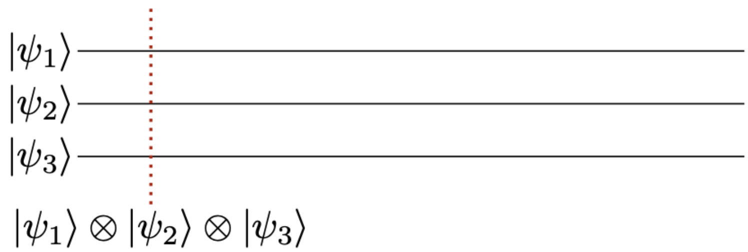
There is no general purpose unitary cloning operator.

- There is **no** quantum gate that copies **any** quantum state using another initialized quantum state.
- If there is a quantum gate that copies a particular quantum state $|\Psi\rangle$, then this operator can only work on other quantum states if they are orthogonal to $|\Psi\rangle$.

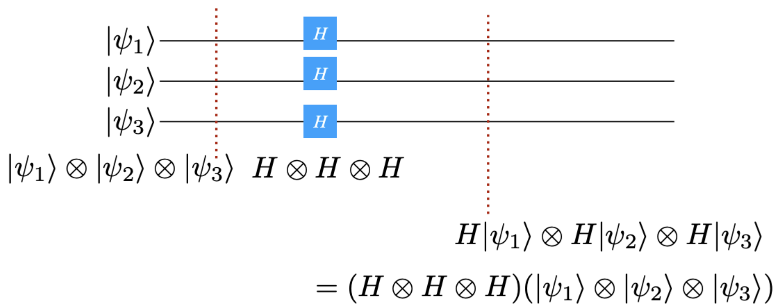
Slide Deck 01C

3-Qubit Quantum Systems, n-Qubit Quantum Systems, and Quantum Teleportation

3-Qubit Quantum Systems, n-Qubit Quantum Systems



The tensor product yields qubit states for more qubits.



Special 3-Qubit States

$$|000\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |001\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \cdots, \quad |111\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Or in other words...

$$|0\rangle^{\otimes n} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \in \quad \mathbb{C}^{2^n}$$

Quantum Teleportation

https://xkcd.com/465/

Teleporting

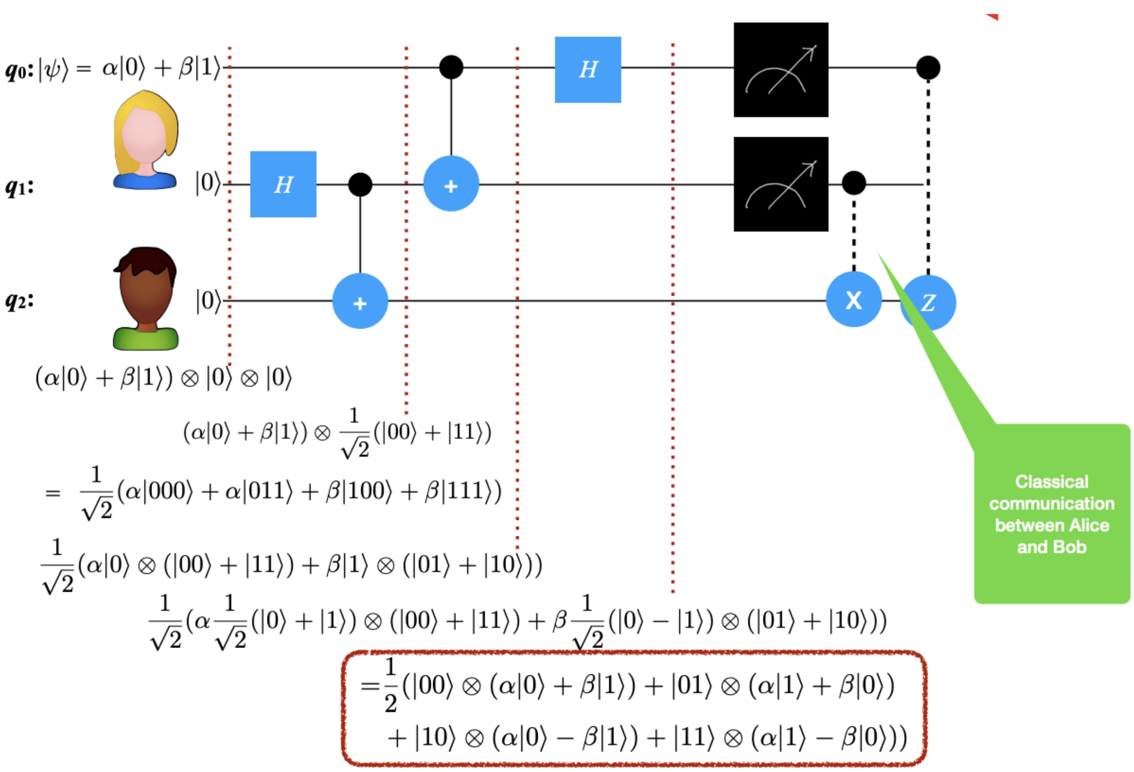
Teleporting $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ from one qubit to another.

- Moving a qubit state from one qubit to another.
- The qubits can be physically distant.
- Supported by classical communication.

Note: $|\psi\rangle$ remains unknown during the process.

Thus, we want to move $|\psi\rangle$ from q_0 to q_2.

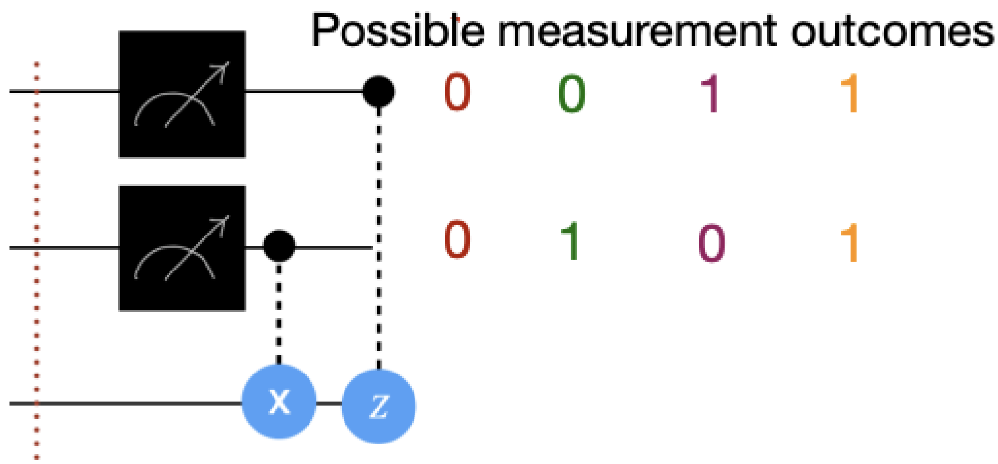
Example



Where, qubits q_1 and q_2 are entangled and prepared such that q_2 can be taken to a distant location.

Measurement

Alice communicates her classical bits obtained by measuring her two qubits to Bob.



$$\frac{1}{2}(|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle)$$
$$+ |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + |11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle))$$

Outcome: $(\alpha|0\rangle + \beta|1\rangle)$

Thus, no matter her measurement, since Bob knows the outcome of Alice' measurement, he can apply corresponding quantum gates to his qubit.

Then he knows that he has obtained exactly the quantum state that Alice originally kept in her first qubit, although he has no idea about the actual values of α and β.

Note: Alice's qubit is destroyed through measurement.

Teleportation

Classical communication necessary to apply correct gate(s) to third qubit for final transformation: fast-than-speed-of-light teleportation not possible.

Depending on measurement result, either X, Z, or both, of just I is applied to remaining term.

Questions

- Why would we want to copy in quantum computing? I'm not sure what is meant by copying.
- Is teleportation only with 3-qubits? What is the proof? (i.e., n+1 qubits).
- Does teleportation collapse the state? Is their partial teleportation
- Is teleportation only with the Hadamard gate?