SENG 480D - Quantum Algorithms and Software Engineering

Theory and implementation of quantum algorithms including challenges and opportunities for quantum software developers and engineers using Qiskit, Q# SDK, Jupyter Notebooks, and Interactive Textbooks.

Lecture 04 Slide Deck:

- lecture-04-slide-deck-01.pdf

Slide Deck 01

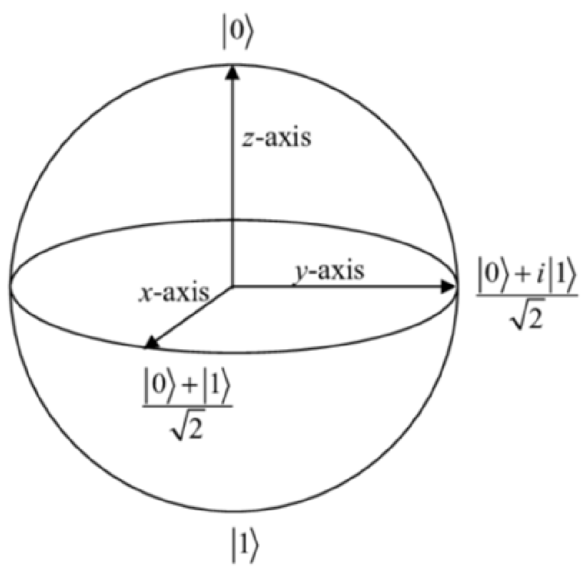
Qubits, States, Bloch Sphere

A physical qubit is a two-level mechanical systems. It can represented as a two-dimensional complex Hilbert space $\mathbb{C}^2$. A qubit state, at any given time, can be represented by a vector in $\mathbb{C}^2$.

$\mathbb{C}^2$ is equipped, by definition, with an inner product which allows us to determine the position of two vectors representing two qubit states. If the two vectors are orthogonal, they are basis vectors $\mathbb{C}^2$.

Ket zero and Ket one are the most famous basis vectors constituting the computational basis.

A sphere with radius 1 (i.e., Bloch Sphere) summarizes the states of a single qubit. Bloch sphere surface states are pure states and can be represented by a linear combination of two basis vectors: $\psi = \alpha|0\rangle + \beta|1\rangle$. The pure states satisfy the Born Rule: $|\psi|^2 = |\alpha|^2 + |\beta|^2 = 1$. States inside the Bloch sphere are called mixed states.



Bloch Sphere with Computational Basis

The basis vectors on the z-axis - kets $|0\rangle$ and $|1\rangle$ - point to the north and south poles of the Bloch sphere, respectively. These states form the Computational basis. Any diagonal in Bloch sphere constitutes a basis. Any point on the Bloch sphere can be expressed as a linear combination of these two basis states.

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

Bloch Sphere with Hadamard or Fourier Basis

Applying the Hadamard or H gate to the two basis ket $|0\rangle$ and $|1\rangle$, results in vectors pointing in opposite directions to the equator along the x-axis.

These two states form the Hadamard or Fourier basis. These kets or states are named $|+\rangle$ and $|-\rangle$. They are also called: Spin up $|\uparrow\rangle = |+\rangle$ and Spin down $|\downarrow\rangle = |-\rangle$.

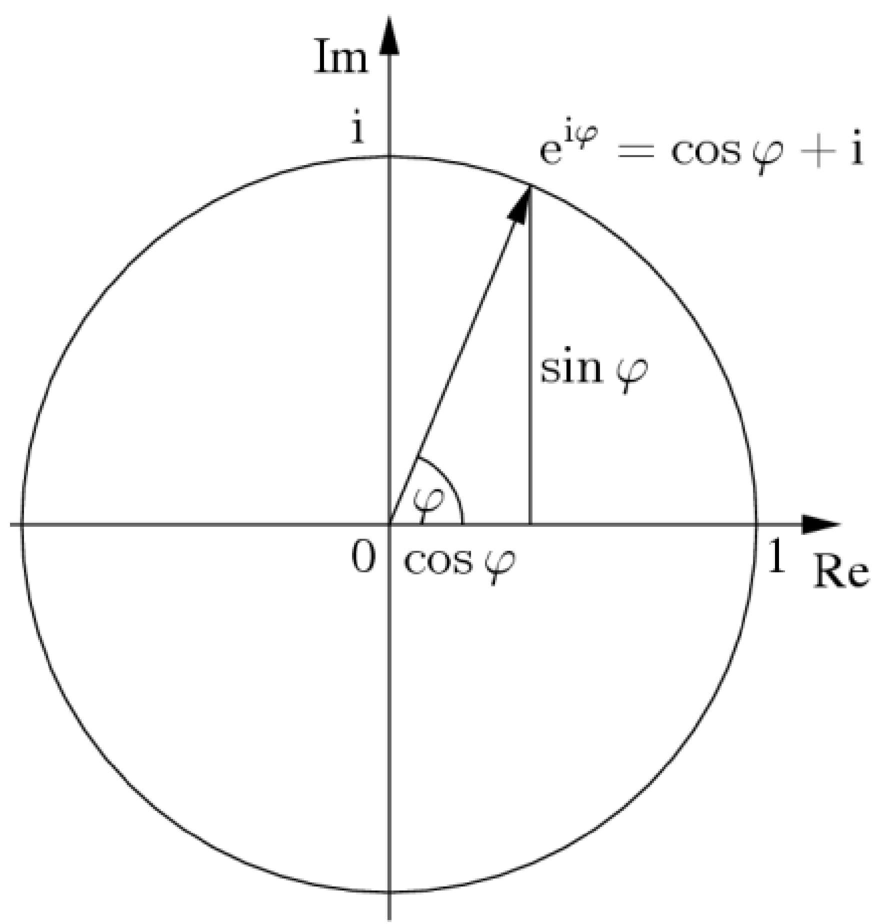
Bloch Sphere with Circular Basis

There are many other points on the equator that can be reached by changing the relative phase - angle against the x-axes.

The points where the y-axes intersects the equator can be expressed with complex numbers. These two states form the Circular basis. These kets are denoted by $|i\rangle$ and $|-i\rangle$ or $|\circlearrowright\rangle = |i\rangle$ and $|\circlearrowleft\rangle = |-i\rangle$.

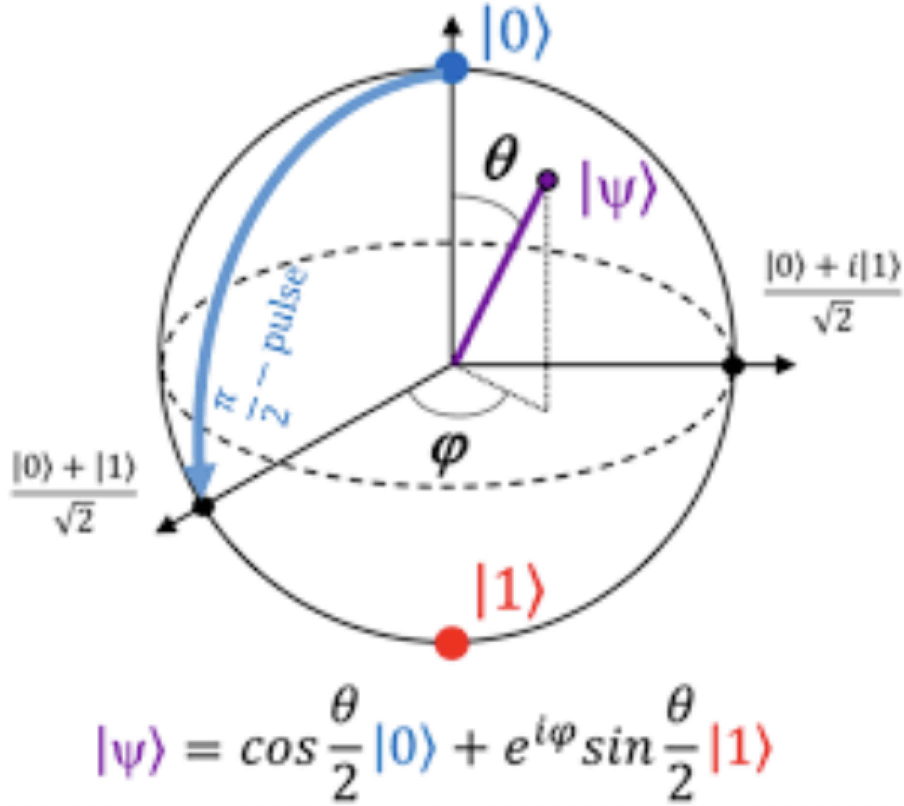
Euler's Formula

Foundational equation in complex arithmetic. Euler's formula establishes the fundamental relationship between trigonometric functions and exponential functions. Geometrically, it can be thought of as a way of bridging two representations of the same unit complex number in the complex plane.



Bloch Sphere with Cartesian and Polar Coordinates

- Cartesian Coordinates: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- Polar Coordinates: $|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$



The state of a qubit can be represented in terms of two parameters ϕ and θ. We have $|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, where $0 \leq \theta \leq \pi$ and $0 \leq \theta \leq 2\pi$.

What is the state of a qubit?

$$\begin{aligned}|\Psi\rangle &= \alpha|0\rangle + \beta|1\rangle, \text{where } \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1\\ &= r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle\\ &= e^{i\phi_0}[r_0|0\rangle + r_1 e^{i(\phi_1-\phi_0)}|1\rangle]\end{aligned}$$

where, $e^{i\phi_0}$ is the global phase that can be ignored and $i(\phi_1 - \phi_0) = \phi$.

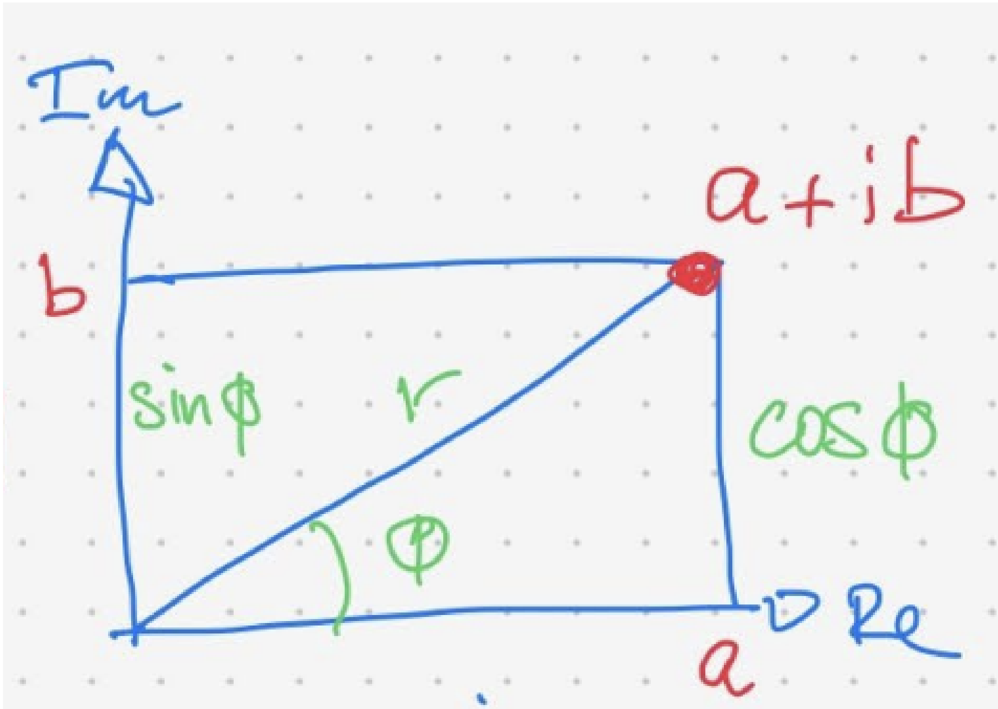
$$r_0 = \cos\frac{\theta}{2} \in \mathbb{R}$$

$$r_1 = \sin\frac{\theta}{2} \in \mathbb{R}$$

$$\text{so, } r_0^2 + r_1^2 = 1$$

Now we have a quantum state with two real parameters (i.e., θ and ϕ) instead of four parameters!

We note that α, and β have each two real parameters. How do we visualize that? We remember that $|0\rangle$ and $|1\rangle$ are orthogonal states.



where,

$$\alpha = a + ib$$
$$= re^{i\phi}$$
$$= r(cos\phi + i\sin\phi).$$

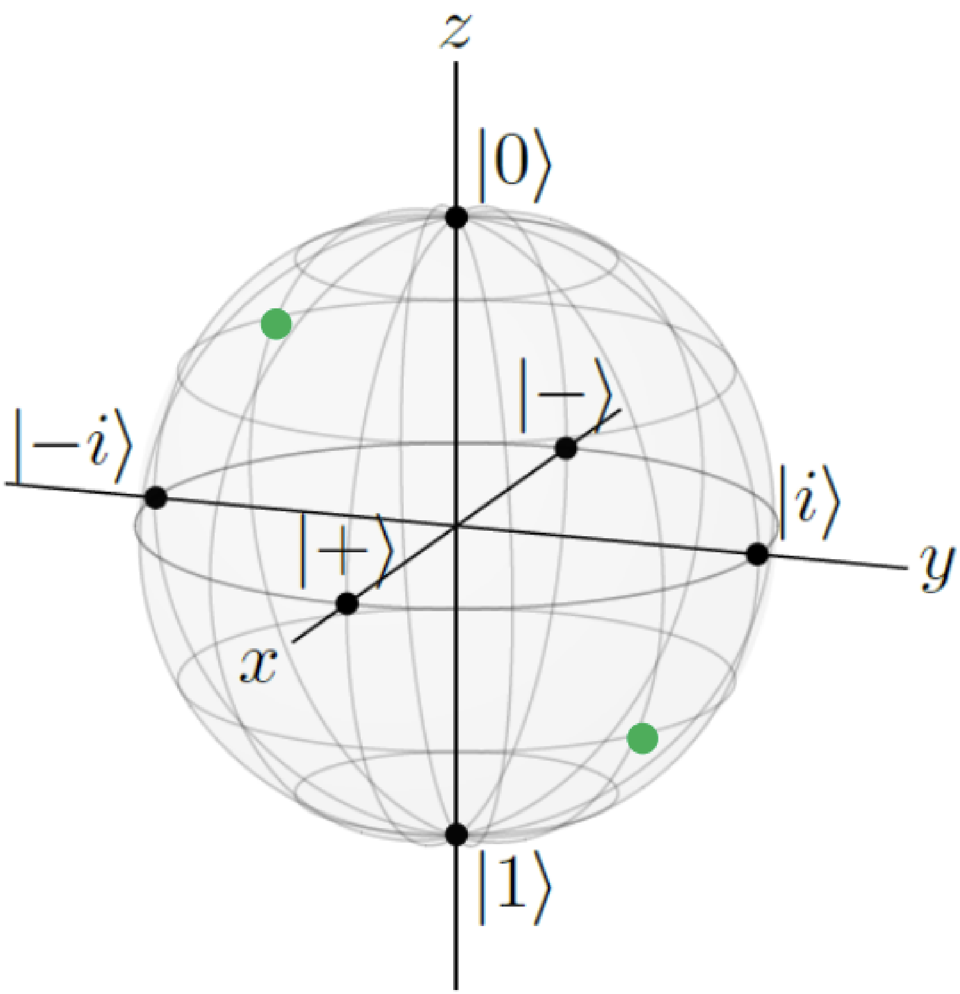
Thus,

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

is a quantum state in 3D space, Bloch sphere, where $|0\rangle$ and $|1\rangle$ are antipodal states in opposite directions.

Bloch Sphere - Basis States

Summary of Basis States: Any two Bloch sphere antipodes are basis states.



Interesting Quiz Questions

- Where are the following three states located on the Bloch Sphere?
- Verify that they satisfy the Born Rule.
- What are the amplitudes of the three states?
- What are the probabilities - the norm-square of the amplitudes - that these states collapse to 0 or 1 when measuring them?

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{\frac{i\pi}{6}}|1\rangle)$$

$$\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

$$\frac{\sqrt{2}}{3}|0\rangle + \frac{1-2i}{3}|1\rangle$$

Measurements

Measurement collapses a qubit state to a classical bit. Determining the state of a qubit is to measure the **projection** of its state vector along a given axis - z-axis, y-axis, x-axis, or arbitrary axis.

Measuring points located in the **northern hemisphere** in z-directions results in $|0\rangle$ and northern hemisphere in $|1\rangle$. After a projection measurement is completed, the qubit will be in either one of its computational basis states. Measuring a quantum state along the z-axis results in $|0\rangle$ or $|1\rangle$ and collapse to 0 or 1, classically.

If the state is expressed as a linear combination of the basis states, then the norm-square amplitudes (i.e., coefficients) are the probabilities that a states collapses to 0 or 1 when measuring.

- Measuring along the z-axis results in $|0\rangle$ or $|1\rangle$ and collapses to 0 or 1, classically.
- Measuring along the x-axis results in $|+\rangle$ or $|-\rangle$ and collapses to 0 or 1, classically.
- Measuring along the y-axis results in $|+i\rangle$ or $|-i\rangle$ and collapses to 0 or 1, classically.

To get meaningful and accurate results, measurements must be repeated many times (e.g., 1000 times).