SENG 480D - Quantum Algorithms and Software Engineering

Lecture 03 Slide Deck:

- lecture-03-slide-deck-01.pdf

Slide Deck 01A

Quantum Computing

1. Describing a quantum system.
2. Describing operations (gates) on a quantum system.
3. Composing multiple qubit quantum systems from smaller ones.
4. Measuring (obtaining classical information from) a quantum system.

Review

Bloch Sphere

Any point on the surface of the Bloch sphere corresponds to a qubit state.

Quantum Algorithms

Qubit states during quantum algorithm: No peaking. Quantum states can not be checked during a computation without collapsing the state to a classical bit value through measurement.

Qubit State

A mathematical representation of a qubit.

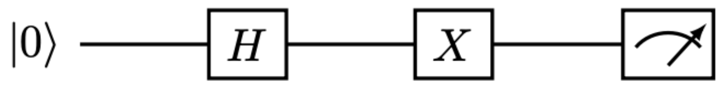
- Superposition: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- Born Rule: $|\alpha|^2 + |\beta|^2 = 1$
- $\alpha, \beta \in \mathbb{C}$

The classical 0 is measure with probability $|\alpha|^2$. The classical 1 is measured with probability $|\beta|^2$.

1-Qubit Gates

- **Pauli Gates**: X-gate (i.e., Quantum Not-Gate), Y-Gate, Z-Gate
- **Hadamard Gate**: H-Gate (Equal Superposition)
- S-Gate (Phase Gate), T-Gate ($\frac{\pi}{8}$ Gate)

1-Qubit Circuits



Quantum Gates Are Unitary Matrices

A matrix U is unitary if $U^{\dagger}U = I$, where I is the identity matrix.

Note: $U^{\dagger}$ is equivalent to U^{*}, U^{H}, and U^{+}. It is specifically complex conjugate (a Hermitian Transpose).

Thus, if U is unitary then...

- $U^{\dagger}U = UU^{\dagger}$
- U is reversible/invertible
- U is diagonalizable
- $\mid det(U) \mid = 1$
- U, applied to a vector, preserves the norm of the vector.

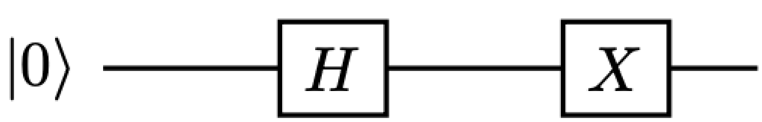
Note: U is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that: $P^{-1}UP = D$.

Quantum Gates

Unitary operation/matrix (requirement for all quantum gate operations).

Single Qubit: Rotation on Surface of Bloch Sphere

Quantum Circuits and Mathematical Descriptions



Dirac Notation: $XH|0\rangle$

Matrix/Vector Notation:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note: We notice that the notation is the reverse order of the quantum circuit (i.e., right to left). We also notice that we do not have to apply $\frac{1}{\sqrt{2}}$ when solving.

Solution 01

Step 1

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (1 \times 1) + (1 \times 0) \\ (1 \times 1) - (1 \times 0) \end{bmatrix}$$

Step 2

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (0 \times 1) + (1 \times 1) \\ (1 \times 1) - (0 \times 1) \end{bmatrix}$$

Conclusion

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution 02

Step 1

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (0 \times 1) + (1 \times 1) & (0 \times 1) - (1 \times 1) \\ (1 \times 1) + (0 \times 1) & (1 \times 1) - (0 \times 1) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Step 2

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (1 \times 1) - (1 \times 0) \\ (1 \times 1) - (1 \times 0) \end{bmatrix}$$

Conclusion

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution 03

Step 1

$$
\begin{aligned}
XH|0\rangle &= X(\frac{1}{\sqrt{2}}|0\rangle \;+\; \frac{1}{\sqrt{2}}|1\rangle) \\
&= \frac{1}{\sqrt{2}}|1\rangle \;+\; \frac{1}{\sqrt{2}}|0\rangle \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}
\end{aligned}
$$

Applying Quantum Gates

Matrix operators serving as quantum gates must be norm preserving.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,\;\; |\alpha|^2 + |\beta|^2 = 1,\;\; \alpha, \beta \in \mathbb{C}$$

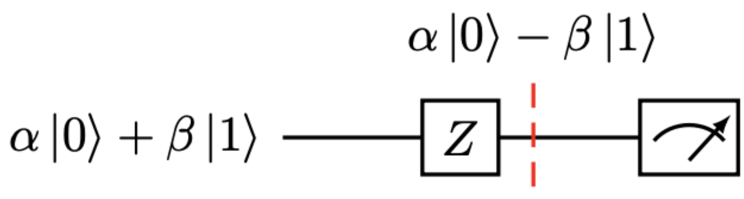
Gates applied to quantum state again must yield quantum state.

$$|\psi\rangle' = \alpha'|0\rangle + \beta'|1\rangle,\;\; |\alpha'|^2 + |\beta'|^2 = 1,\;\; \alpha', \beta' \in \mathbb{C}$$

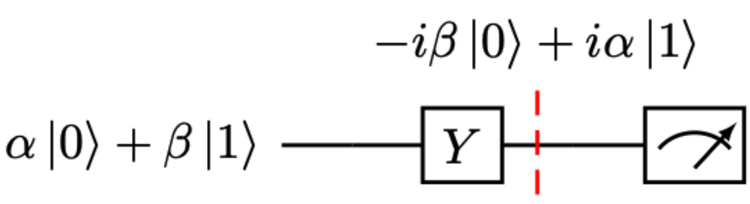
Quantum Algorithm Output: Measure

Once a qubit is measured the qubit state is destroyed, only classical information (i.e., 0 or 1) remains.

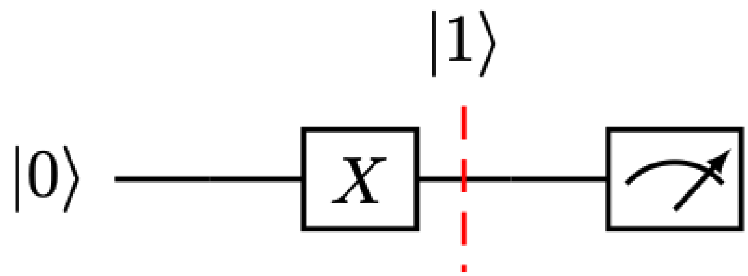
$Z(\alpha|0\rangle + \beta|1\rangle)$ = $\alpha|0\rangle - \beta|0\rangle$



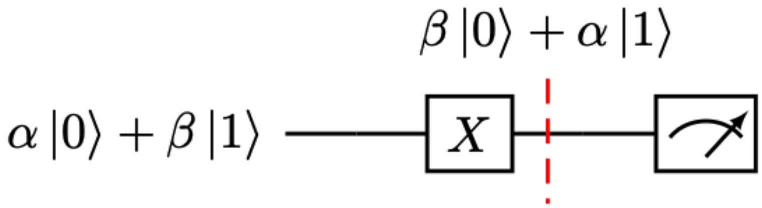
$Y(\alpha|0\rangle + \beta|1\rangle)$ = $-i\beta|0\rangle + i\alpha|1\rangle$, measuring 0: $|-i\beta|^2 = |\beta|^2$



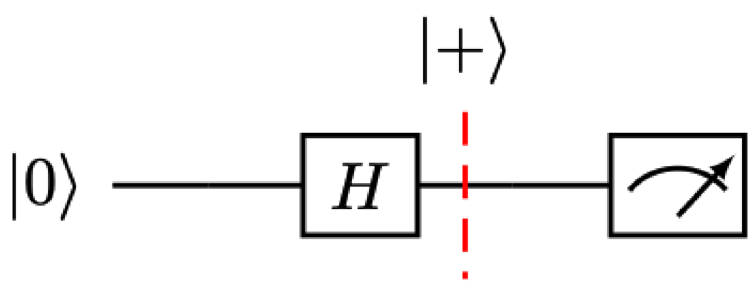
$X|0\rangle = |1\rangle$, $\alpha = 0, \beta = 1$, $|\beta|^2 = 1$



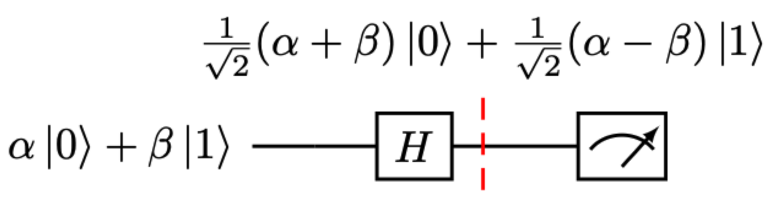
$X(\alpha|0\rangle + \beta|1\rangle)$ = $\beta|0\rangle + \alpha|1\rangle)$



$H|0\rangle$ = $|+\rangle$, $\alpha = \frac{1}{\sqrt{2}}, |\alpha|^2 = \frac{1}{2}$ and $\beta = \frac{1}{\sqrt{2}}, |\beta|^2 = \frac{1}{2}$



$H(\alpha|0\rangle + \beta|1\rangle)$ = $\frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle$



Note: Measurements project onto computational basis.

Slide Deck 01B

2-Qubit Quantum Systems

- **Superposition**
- **Entanglement**
- **Measurement**

It consists of two qubits instead of just one.

Computational Basis

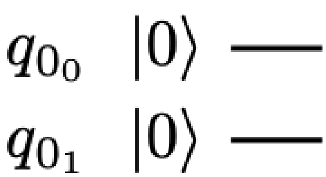
$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$|10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \qquad |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

2-Qubit Quantum State

Example

$|00\rangle$ = $|0\rangle \otimes |0\rangle$



Solution

$$|0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1\\0\end{bmatrix}\\0\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Remember: In general $A \otimes B \neq B \otimes A$.

Properties of Tensor Product

Operation to form larger vector spaces (from smaller ones).

Given kets $|\phi\rangle$, $|\psi\rangle$, $|\rho\rangle$, and $|\sigma\rangle$, and $r \in \mathbb{R}$. Then...

- $r(|\phi\rangle \otimes |\psi\rangle) = (r|\phi\rangle) \otimes |\psi\rangle = |\phi\rangle \otimes (r|\psi\rangle)$
- $(|\phi\rangle + |\psi\rangle) \otimes |\rho\rangle = |\phi\rangle \otimes |\rho\rangle + |\psi\rangle \otimes |\rho\rangle$
- $|\rho\rangle \otimes (|\phi\rangle + |\psi\rangle) = |p\rangle \otimes |\phi\rangle + |\rho\rangle \otimes |\psi\rangle$
- $(|\phi\rangle \otimes |\psi\rangle)^\dagger = |\phi\rangle^\dagger \otimes |\psi\rangle^\dagger = \langle\phi| \otimes \langle\psi|$
- $(\langle\phi| \otimes \langle\psi|)(|\rho\rangle \otimes |\sigma\rangle) = \langle\phi|\rho\rangle \langle\psi|\sigma\rangle$

Given linear operators A and B and kets $|\phi\rangle$ and $|\psi\rangle$...

- $(A \otimes B)(|\phi\rangle \otimes |\psi\rangle) = A|\phi\rangle \otimes B|\psi\rangle$

2-Qubit States

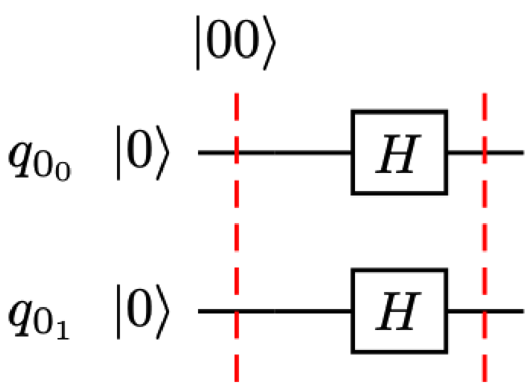
We have seen 2-qubit states that we described as tensor products BUT not every 2-qubit state can be described as a tensor product.

$$|\psi\rangle = |\psi_1\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

2-Qubit State in Superposition

$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$ with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$

Example



Solution 01

$$\begin{aligned}
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned}$$

Solution 02

$$
\begin{aligned}
H|0\rangle \otimes H|0\rangle &= (H \otimes H)|00\rangle \\
&= (\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix})(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}) \\
&= \frac{1}{2}\begin{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} & \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\\ \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} & -\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\end{bmatrix}\begin{bmatrix}1 & \begin{bmatrix}1\\0\end{bmatrix}\\ 0 & \begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned}
$$

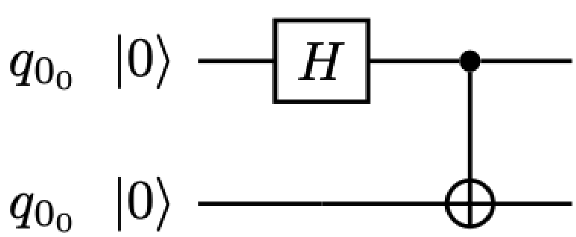
Slide Deck 01C

Entanglement

Entanglement: A pair of qubits is **entangled** when the quantum state of each cannot be described independently of the quantum state of the other.

Two qubits can be entangled using H- and C_X-gates:

Example: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

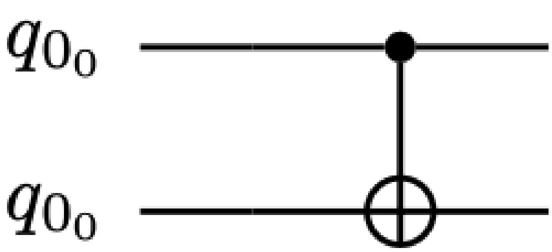


Note: A quantum state is **entangled** if it is not a tensor product.

Controlled NOT Gate / C_X-Gate

2-Qubit Gate

Qubit q_{0_0} is the control qubit and Qubit q_{0_1} is the target qubit.



$$C_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note: The matrix of red numbers is the Identity Matrix and the matrix of green numbers is the X Matrix.

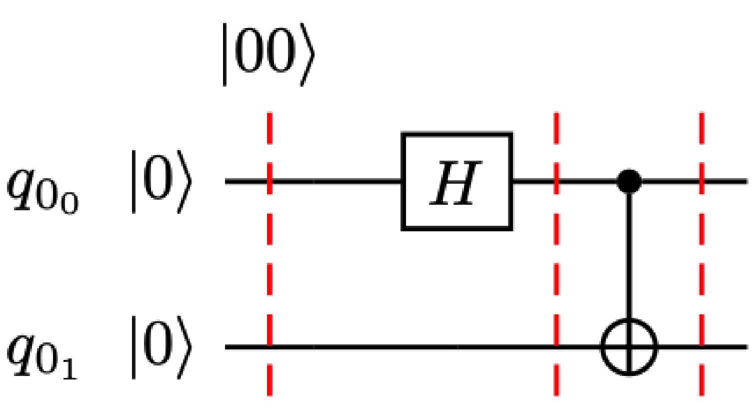
Input $\lvert\psi\rangle$	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 11\rangle$
$C_X\lvert\psi\rangle$	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 11\rangle$	$\lvert 10\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha + 0 + 0 + 0 \\ 0 + \beta + 0 + 0 \\ 0 + 0 + 0 + \gamma \\ 0 + 0 + \delta + 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$$

We can also do the following...

$$\begin{aligned} &C_X(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) \\ &= \alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle \\ &= \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle \end{aligned}$$

Example



- Second Slice: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$
- Third Slice: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$C_X(H \otimes I)|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

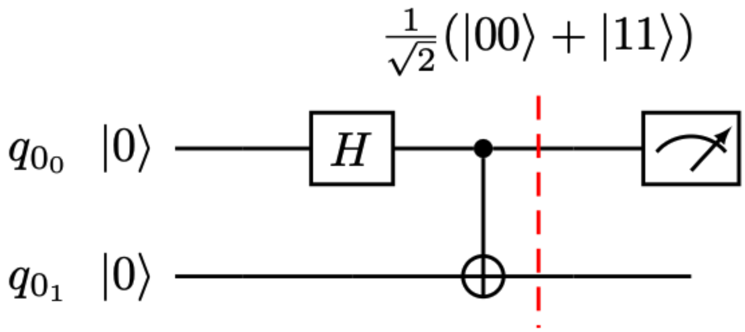
$$= \frac{1}{\sqrt{2}} (\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})$$

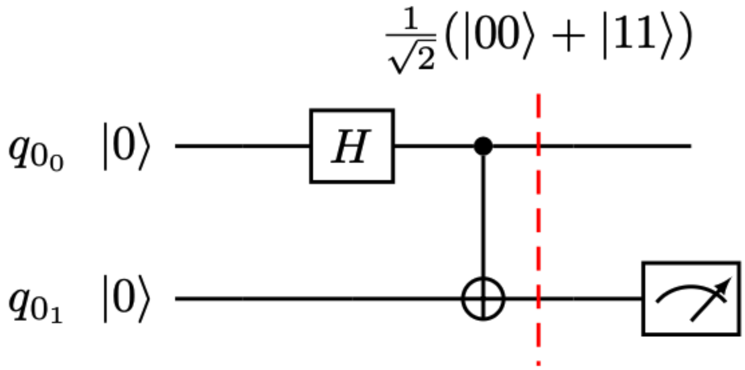
$$= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

Entanglement

Measuring One Qubit: Automatically, instantly also decides information of other (entangled, and possibly distant) qubit.

Example





Solution

If measuring the first qubit results in a 0, then the second qubit is revealed instantly as a 0.

If measuring the first qubit results in a 1, then the second qubit is revealed instantly as a 1.

Note: A quantum state is **entangled** if it is not a tensor product.

> The paradox is only a conflict between reality and your feeling of what reality 'ought to be'. - Richard Feynman

Bell States

There are four famous entangled states called the Bell states.

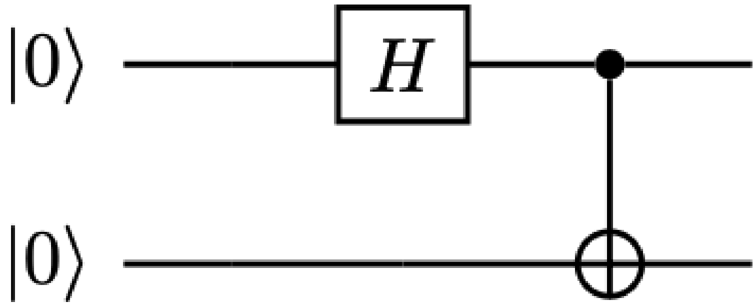
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

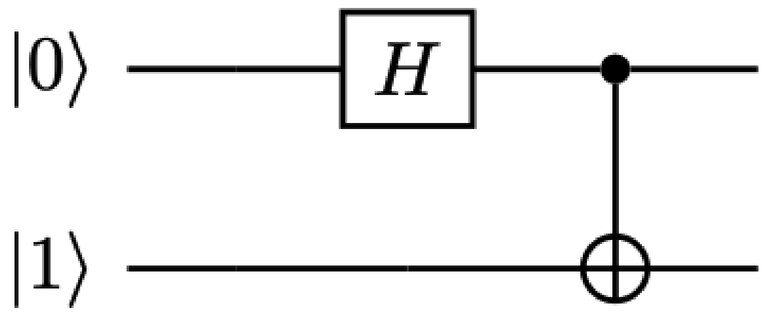
Quantum states consisting of two qubits that represent the most basic examples of quantum entanglement.

States

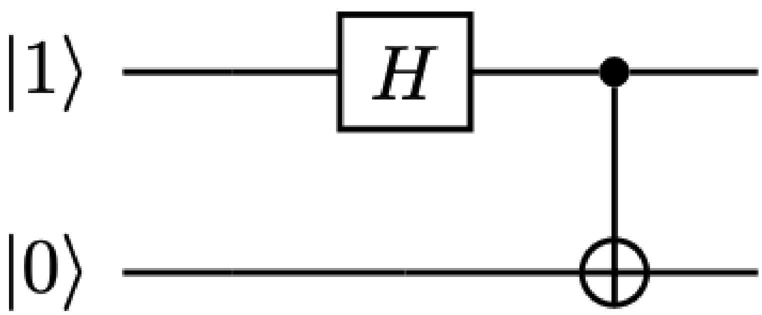
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$



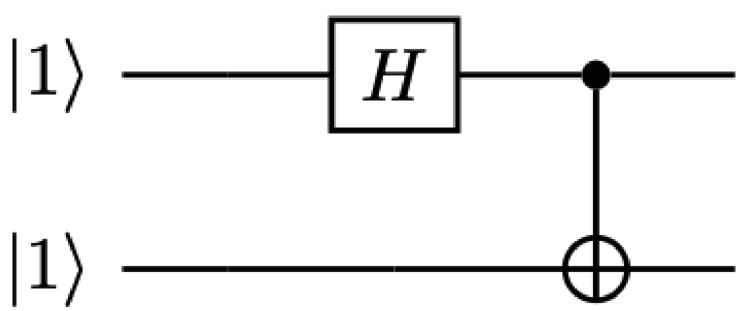
$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle$



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^-\rangle$



$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^-\rangle$



Note: Measuring one qubit of a Bell state instantly reveals information of other qubit - in the form of a classical 0 or 1.

Bell Basis

- The four Bell states build an (orthonormal) basis for $\mathbb{C}^4$:
  - $\langle \Phi^+ | \Phi^- \rangle$, $\langle \Psi^+ | \Psi^- \rangle$
  - $\langle \Phi^+ | \Psi^+ \rangle$, $\langle \Phi^+ | \Psi^- \rangle$
  - $\langle \Phi^- | \Psi^+ \rangle$, $\langle \Phi^- | \Psi^- \rangle$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Questions

- Can the target qubit be the top and the control qubit be the bottom (i.e., bottom up instead of top down)? - Slide 32
- Is it impossible or yet to be solved? - Slide 33
- Is entanglement only in pairs?
- What are shots in quantum probability?

Resources

Quantum Circuits in LaTeX:
https://mirrors.ibiblio.org/CTAN/graphics/pgf/contrib/quantikz/quantikz.pdf