SENG 480D - Quantum Algorithms and Software Engineering

Theory and implementation of quantum algorithms including challenges and opportunities for quantum software developers and engineers using Qiskit, Q# SDK, Jupyter Notebooks, and Interactive Textbooks.

Lecture 02 Slide Deck:

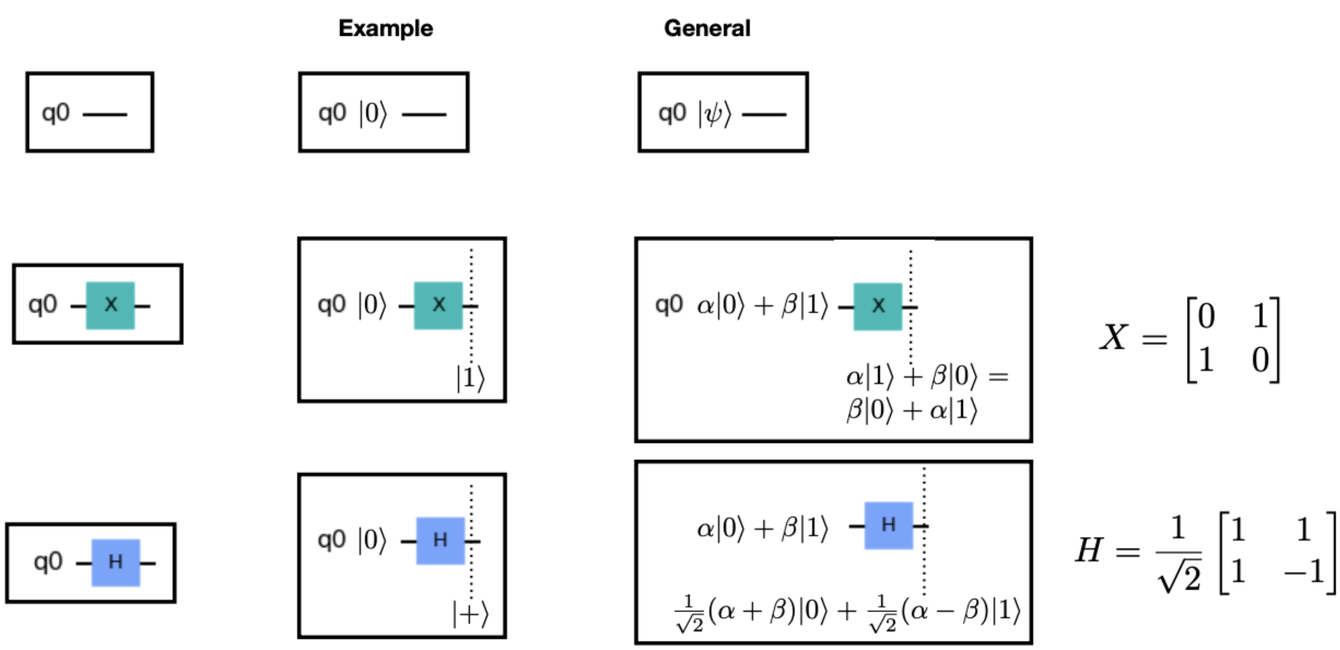
- lecture-02-slide-deck-01.pdf
- lecture-02-slide-deck-02.pdf

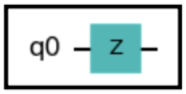
Slide Deck 01

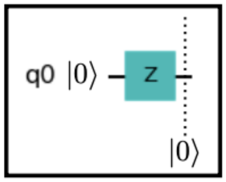
Quantum Circuit

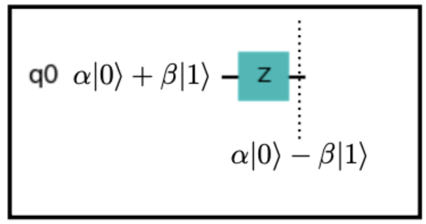
Computational routine consisting of coherent quantum operations on qubits: ordered sequence of quantum gates, measurements (and resets).

1-Qubit Quantum Systems: Circuits and Gates

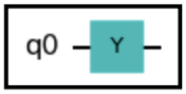


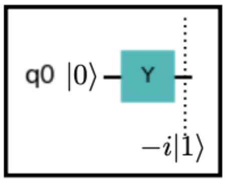


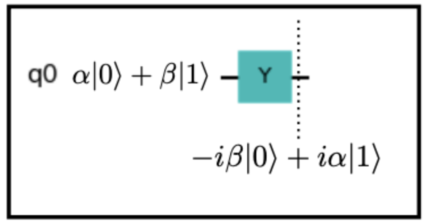




$$Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$







$$Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

Matrices

Matrices sometimes serve as quantum (logic) gates. They are linear operators.

$$M_{n,m}(\mathbb{C}) : \text{set of all n x m complex matrices}$$

Note: Quantum gates are (n x n) matrices.

$$A \in M_{n,n}(\mathbb{C}) : \quad a_{ij} \in \mathbb{C}$$

Matrix Operations

- **Multiply by Scalar:** $\alpha A_{ij} = (\alpha A)_{ij}$
- **Addition:** $A_{ij} + Bij = (A + B)_{ij}$
- **Multiply Matrices:** $A \ x \ B$
- **Tensor Product:** $A \bigotimes B$
- **Complex Conjugate** A^*: $(A^*)_{ij} = (A_{ij})^*$
- **Transpose** A^T: $(A^T)_{ij} = A_{ji}$
- **Conjugate Transpose** $A^\dagger$: $(A^\dagger)_{ij} = A^*_{ji}$
- **Inverse** A^{-1}: $AA^{-1} = I$

Note: Matrices are not commutative. In general A, B: n x n matrices. $AB \neq BA$.

Properties of Matrix Multiplication

Let $A, B, C \in M_{n,n}(\mathbb{C})$...

- (AB)C = A(BC) = ABC
- A(B + C) = AB + AC
- (A + B)C = AC + BC

- λ(AB) = (AB)λ = (λA)B = A(λB), $\lambda \in \mathbb{C}$
- (AB)* = A*B*, (AB)^T = B^{T}A^T, (AB)† = B†A†

Vectors

Vectors are special cases of matrices.

Multiply by Scalar:

$$\alpha \begin{bmatrix} b_1 \\ b_n \end{bmatrix} = \begin{bmatrix} \alpha b_1 \\ \alpha b_n \end{bmatrix}$$

Addition:

$$\begin{bmatrix} b_1 \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 + c_1 \\ b_n + c_n \end{bmatrix}$$

Applying Quantum Gates

Matrix operators serving as quantum gates: norm preserving. In other words, quantum gates are norm preserving (n x n) matrices.

Quantum gates are unitary matrices.

Unitary Matrices and Properties

A matrix U is unitary if $U^\dagger U = I$, where I is the identity matrix.

If U is unitary then...

- $U^\dagger U = UU^\dagger$
- $U^\dagger = U^-1$
- U is reversible
- U is diagonalizable
- $|det(U)| = 1$
  - U, applied to a vector, preserves the norm of the vector.

Note:

$$det(U) = \sum_{j=1}^{n} (-1)^{i+j} U_{ij} M_{ij}$$

where U_{ij} is the element in U at position (i, j) and M_{ij} is the determinant of the matrix obtained by removing from U the i-th row and j-th column.

X-Gate

The NOT-Gate/ X-Gate is usually represented by an X or + sign.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore...

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (0)(\alpha) + (1)(\beta) \\ (1)(\beta) + (0)(\alpha) \end{bmatrix}$$

Remember: Multiplying a matrix by a vector.

$$\begin{bmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_n \end{bmatrix} = \begin{bmatrix} (a_{11})(a_1) + (a_{1n})(a_n) \\ (a_{21})(a_1) + (a_{2n})(a_n) \end{bmatrix}$$

- Input $|0\rangle$ apply X and obtain $|1\rangle$.
- Input $|1\rangle$ apply X and obtain $|0\rangle$.
- Input $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ apply X and obtain $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$.

In other words...

$$\begin{aligned} X(\alpha|0\rangle + \beta|1\rangle) &= \alpha X|0\rangle + \beta X|1\rangle \\ &= \alpha|1\rangle + \beta|0\rangle \\ &= \beta|0\rangle + \alpha|1\rangle \end{aligned}$$

Z-Gate

The Z-Gate is represented by a Z sign.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore...

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha + 0 \\ 0 - \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

- Input $|0\rangle$ apply Z and obtain $|0\rangle$.
- Input $|1\rangle$ apply Z and obtain $-|1\rangle$.
- Input $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ apply X and obtain $\begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$.

In other words...

$$\begin{aligned} Z(\alpha|0\rangle + \beta|1\rangle) &= \alpha Z|0\rangle + \beta Z|1\rangle \\ &= \alpha|0\rangle - \beta|1\rangle \end{aligned}$$

Y-Gate

The Y-Gate is represented by a `Y` sign.

$$Z = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Therefore...

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (0)(\alpha) - (i)(\beta) \\ (i)(\alpha) - (0)(\beta) \end{bmatrix} = \begin{bmatrix} -(i)(\beta) \\ (i)(\alpha) \end{bmatrix}$$

- Input $|0\rangle$ apply Y and obtain $i|1\rangle$.
- Input $|1\rangle$ apply Y and obtain $-i|0\rangle$.
- Input $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ apply Y and obtain $\begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}$.

In other words...

$$= -i\beta|0\rangle + i\alpha|1\rangle$$

Hadamard Gate

The Hadamard Gate is represented by an `H` sign.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Therefore...

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

- Input $|0\rangle$ apply H and obtain...

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1+0 \\ 1-0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \end{aligned}$$

- Input $|1\rangle$ apply H and obtain...

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0+1\\0-1\end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Input $\begin{bmatrix}\alpha\\\beta\end{bmatrix}$ apply H and obtain...

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\\alpha-\beta\end{bmatrix}$$

In other words...

$$\begin{aligned}H(\alpha|0\rangle + \beta|1\rangle) &= \alpha H|0\rangle + \beta H|1\rangle\\ &= \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle = |1\rangle)\\ &= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle\end{aligned}$$

Pauli-Gates

X, Y, and Z - gates: Pauli Gates

Quantum Gates

Unitary operation / matrix (requirement for all quantum gates!).

Single Qubit: Rotation on surface of Bloch Sphere.

Cancellation of Gate Operations

While all quantum gates are reversible (i.e., $U^\dagger U = I$) some quantum gates even cancel themselves when applied twice in a row.

- I

$$II = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

- X

$$XX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0+1 & 0+0\\0+0 & 1+0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

- Z

$$ZZ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- H

$$HH = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$$

Matrices where $U^\dagger = U$ are called Hermitian.

Note: Given matrix M, we show $MM = I$.

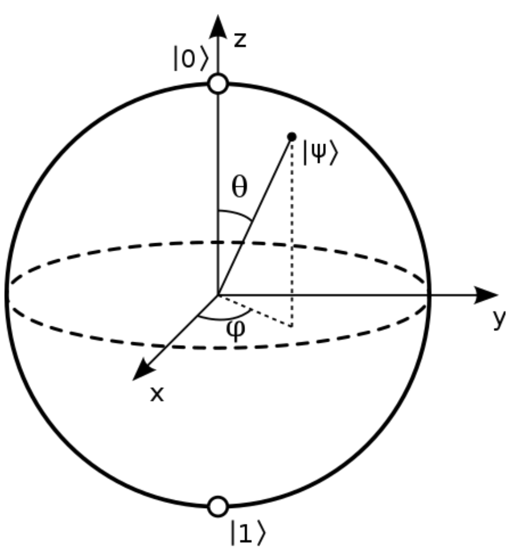
Phase Shift Gates

$$P(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

If $\phi = \frac{\pi}{4}$: T-Gate; $P(\frac{\pi}{4})$ = T.

If $\phi = \frac{\pi}{2}$: S-Gate; $P(\frac{\pi}{2})$ = S

Note: $e^{i\frac{\pi}{2}} = i$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.



Important Basis Vectors

A basis consist of a set or pairwise orthogonal vectors.

- Computational Basis States: $|0\rangle$ and $|1\rangle$

$$\langle\, 0 \,|\, 1 \,\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1)(0) + (0)(1) = 0$$

H rotates qubit on surface of Bloch Sphere.

- $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- $|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\langle + \mid - \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = (\frac{1}{\sqrt{2}})^2 - (\frac{1}{\sqrt{2}})^2 = 0$$

Kets: Inner Product and Outer Product

Inner Product (Dot Product)...

$$\langle \psi \mid \phi \rangle = \sum_{i=1}^{d} \alpha_i^* \, \beta_i^*$$

where $\langle\psi|$ is the row vector and $|\phi\rangle$ is the column vector.

$\langle \psi \mid \phi \rangle = 0$ if and only if $|\psi\rangle$, $|\phi\rangle$ are orthogonal, where...

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \ |\phi\rangle = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

Outer Product...

$$\mid \psi \rangle \langle \phi \mid = \begin{bmatrix} \alpha_1\beta_1^* & \alpha_1\beta_2^* & \dots & \alpha_1\beta_d^* \\ \vdots & \vdots & & \vdots \\ \alpha_d\beta_1^* & \alpha_d\beta_2^* & \dots & \alpha_d\beta_d^* \end{bmatrix}$$

where $|\psi\rangle$ is the column vector and $\langle\phi|$ is the row vector.

Quantum Algorithm Output: Measure

Once a qubit is measured the qubit state is destroyed, only classical information (i.e., 0 or 1) remains.