CSC 320 - Lecture 18

#reduction #np-hard #np #np-complete #3sat #clique #independent-set #dominating-set #graph #polynomial-time-reducibility

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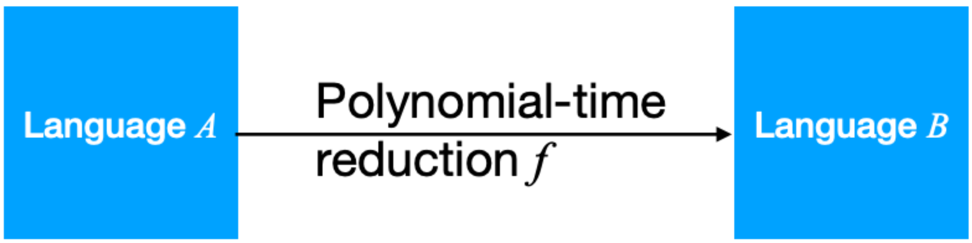
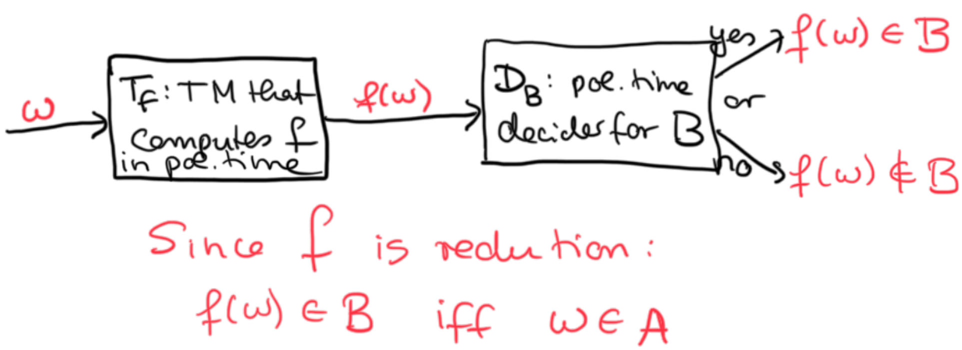
Polynomial-Time Reducibility

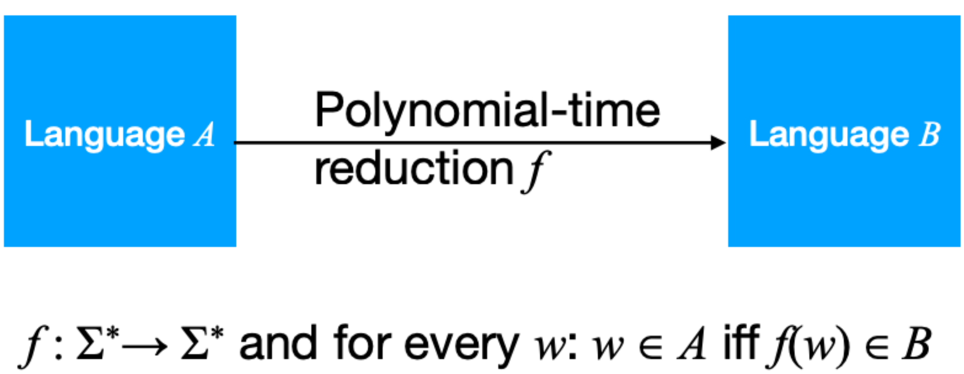


If there is a polynomial time decider D_B for language B then we can answer in polynomial time "$w \in A$?" as follows...



Polynomial time decider for A.

$f(w) \in B \longrightarrow$ YES and $f(w) \notin B \longrightarrow$ NO.



Note. A cannot be harder than B (wrt polynomial-time solvability). Since we can solve A by solving B as a subroutine.

Note. We basically show that any language in NP can be reduced to the satisfiability problem.

If B is NP-Complete and $B \in P$ then $P = NP$

Proof. $B \in NP$ and B is NP-Hard.

That Is. $B \in NP$ and for every $A \in NP: \ A \leq_p B$.

Then. If $B \in P$ and for every $A \in NP$, $A \leq_p B$ then $AinP$.

If B is NP-Complete and $B \leq_p C$ for $C \in NP$, then C is NP-Complete

- Cook/Levin Theorem shows that SAT in NP-Complete.
- The proof works for $3SAT$, showing $3SAT$ is NP-Complete.

NP-Complete Problems

- SAT
- $3SAT$
- $CLIQUE$ (3$SAT \leq_p CLIQUE$)
- Independent Set ($CLIQUE \leq_p IS$)
- Vertex Cover ($IS \leq_p VC$)
- Dominating Set ($VC \leq_p DOMINATING \ SET \ (DS)$)

$CLIQUE \leq_p IS$

$\langle G, k \rangle \in Clique$ if ex. $C \subseteq V$, $|C| = k$, and f.a. $x, y \in C$, $(x, y) \in E$. for $G = (V, E)$

$\langle G', k' \rangle \in IS$ if ex. $I \subseteq V'$, $|I| = k$, and f.a. $a, b \in I$, $(a, b) \in E$. for $G' = (V', E')$

Description of Reduction. For input $\langle G, k \rangle$ for Clique, we build input $\langle G', k' \rangle$ for IS as follows...

$$V' = V$$
$$E' = (V \times V) \setminus E$$

G' has exactly these edges that don't exist in G.

$$k' = k$$

Show.

1. If $\langle G, k\rangle \in Clique$ then $\langle G', k'\rangle \in IS$
2. If $\langle G', k'\rangle \in IS$ then $\langle G, k\rangle \in Clique$

If $\langle G, k\rangle \in Clique$ then $\langle G', k'\rangle \in IS$

Note. $G' = (V, (V \times V) \setminus E);\ k' = k$.

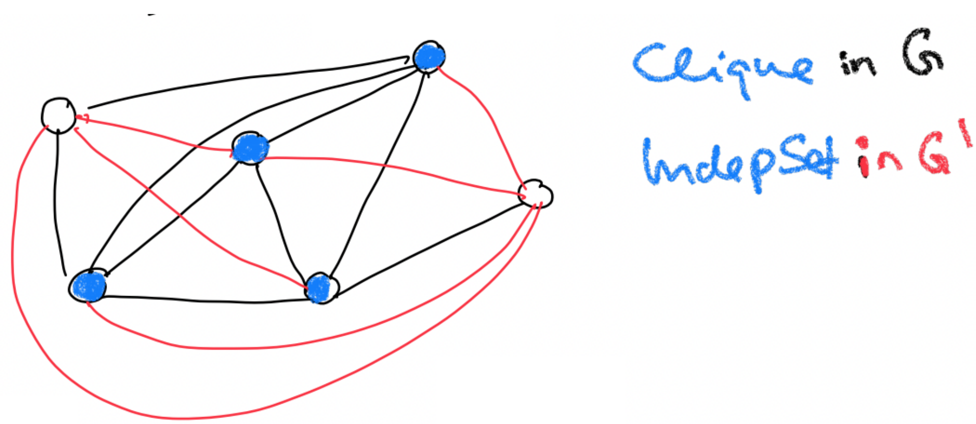
Let $C \subseteq V,\ |C| = k$, be a clique in G. We show that C is an independent set in G'.

Pick any $x, y \in C$. Since C is clique for G $(x, y) \in E \Rightarrow (x, y) \notin E' \Rightarrow$ no edges between vertices of C exist in $G' \Rightarrow C$ is independent set for G'!

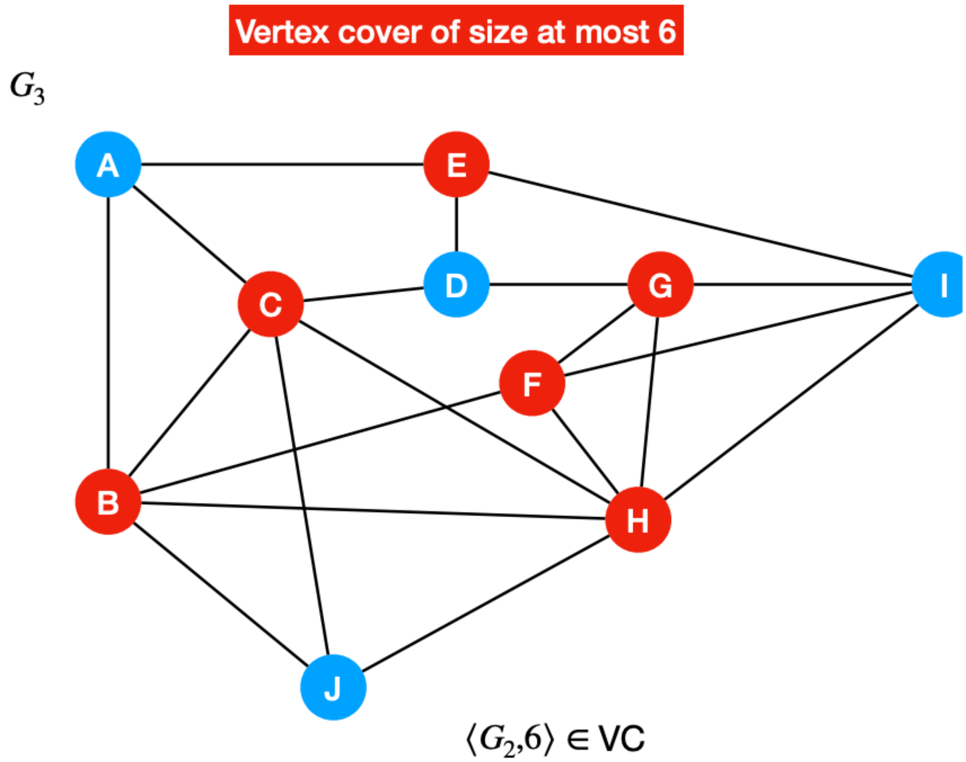
If $\langle G', k'\rangle \in IS$ then $\langle G, k\rangle \in Clique$

Let $I \subseteq V$ be an independent set for G! Then for $x, y \in I,\ (x, y) \notin E' = (V \times V) \setminus E$ $\Rightarrow (x, y) \in E \Rightarrow$ f.a. $a, b \in I$ and $(a, b) \in E \Rightarrow I$ is a clique for G.

Example Graph



$VC \leq_p DS$



Recall. Vertex Cover.

Define Dominating Set (DS)

Note. Midterm question $DS \in NP$.

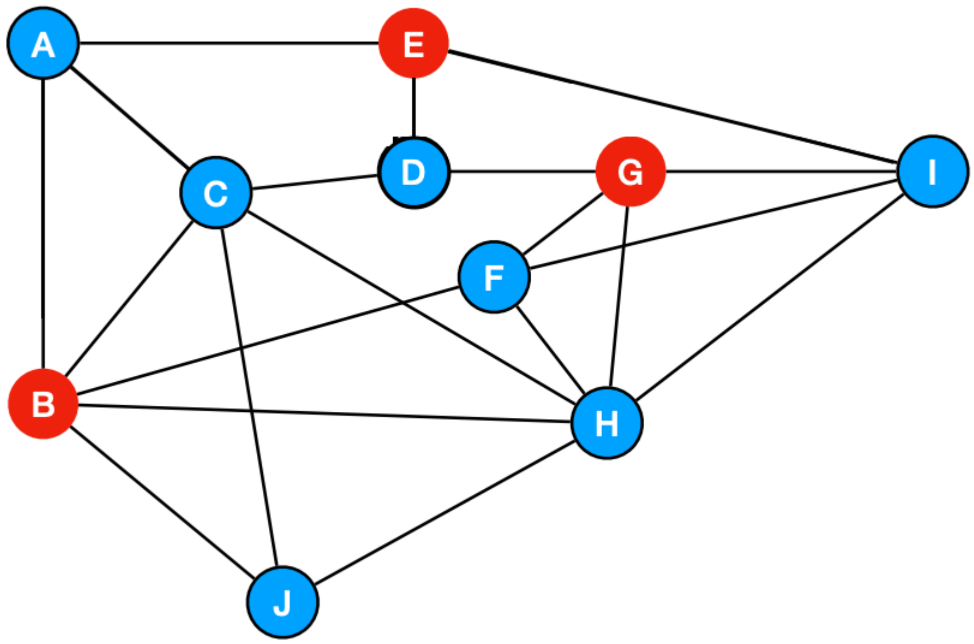
An Interesting Application for Vertex Cover

- Conflict Resolution...
  - Edges whenever two items (vertices) are in conflict to each other, such as time conflicts (edges) when scheduling items (vertices).
- Vertex Cover: items to remove such that the remaining vertex set is conflict free.
- Smallest vertex cover would be the minimum number of items to remove to make data set conflict free.

More Examples of Famous Graph Problems

- **Dominating Set**: An undirected graph $G = (V, E)$ has k-**dominating set** if there exists $V' \subseteq V,\ |V'| = k$ such that for all $v \in V:\ v \in V'$ or v has a neighbour (adjacent vertex) w in G with $w \in V'$.

- $DOMSET = \{\langle G, k\rangle \mid \text{has a dominating set of size at most } k\}$.



A Fun Application for Dominating Set

- Coverage
  - Vertices corresponds to city blocks.
  - Edges connect neighbouring city blocks.
- Place only as many ice cream stand at city blocks such that for each block: the next ice cream stand is at most one block away.

Previous Lecture

[Lecture17]

Next Lecture

[Lecture19]