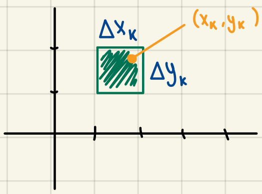
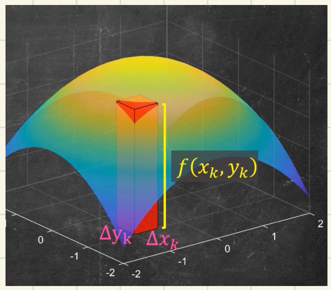
15.1 – Double Integration

- Define double integration of functions over rectangle regions.

Volume Under A Surface





1. Partition the region $[a,b] \times [c,d]$ into little rectangles

$$\Delta A_k = \Delta x_k \Delta y_k$$

2. Choose a point (x_k, y_k) in each rectangle

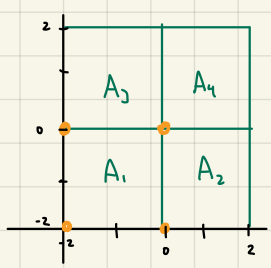
3. Volume $\approx \sum_{k=1}^{n} f(x_k, y_k) \Delta x_k \Delta y_k$

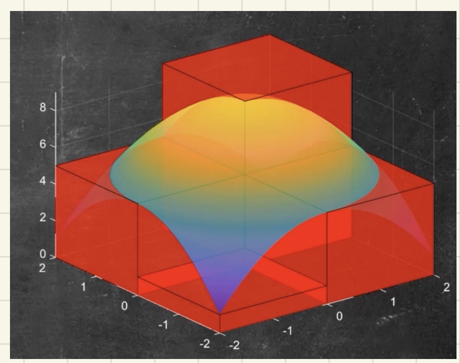
4. Volume $= \lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta x_k \Delta y_k$

Largest Rectangle going to zero area!

Ex. Volume under $9 - x^2 - y^2$ above $[-2,2] \times [-2,2]$

- Use four equal regions.
- Choose (x_k, y_k) to become left point of k^{th} square.





$$\text{Volume} \approx \sum_{k=1}^{4} f(x_k, y_k) \Delta x_k \Delta y_k$$

$$= \sum_{k=1}^{4} f(x_k, y_k)\, 2^2$$

$$= [f(-2,-2) + f(0,-2) + f(-2,0) + f(0,0)]\, 2^2$$

$$= [1 + 5 + 5 + 9]\, 2^2 = 80$$

- Construction of double integrals
- Use of Iterated integrals to compute double integrals
- Finding volume by double integrals

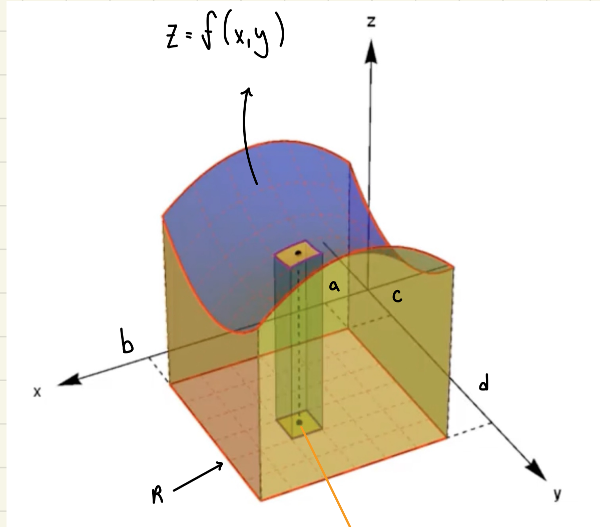
Double and Iterated Integrals over Rectangles

The height of the green tower

$\approx f(x_k, y_k)$

$V_k \approx f(x_k, y_k) \cdot \Delta x_k \Delta y_k$

$\text{Volume} \approx \sum_{k=1}^{n} f(x_k, y_k) \underbrace{\Delta x_k \Delta y_k}_{\Delta A_k}$



R: $a \le x \le b$; $c \le y \le d$

(x_k, y_k) Δy_k Δx_k

$$\boxed{\text{Volume} = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k}$$

$$\boxed{\text{Volume} = \int_R \int f(x,y)\, dA}$$

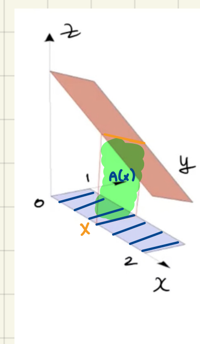
Example: Double integrals

Find the volume under the plane $z = 4 - x - y$ over the region R: $0 \le x \le 2$, $0 \le y \le 1$

$\text{Area} = A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy$

$= \left(4y - xy - y^2/2\right)_0^1$

$\Rightarrow A(x) = 7/2 - x$



$z = 4 - x - y$

$\text{Required volume} = V = \int_{x=0}^{x=2} A(x)\, dx$

$= \int_0^2 (7/2 - x)\, dx$

$= 5$

$$\boxed{\text{Required volume} = 5}$$

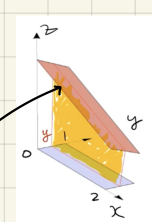
Example: Double integrals

Find the volume under the plane $z = 4 - x - y$ over the region R: $0 \le x \le 2$, $0 \le y \le 1$

$A(y) = \int_{x=0}^{x=2} (4 - x - y)\, dx \quad = 6 - 2y$

$= \left(4x - x^2/2 - xy\right)_0^2$

$z = 4 - x - y$



Volume $= \int_{y=0}^{y=1} A(y)\, dy$

$= \int_0^1 (6-2y)\, dy$

$\boxed{\text{Volume} = 5}$

<u>Note:</u>

$V = \int_0^2 \int_0^1 (4-x-y)\, dy\, dx$ → Iterated Integral!

$V = \int_0^1 \int_0^2 (4-x-y)\, dx\, dy$ → Iterated Integral!

<u>Theorem</u>: Fubini's Theorem (First Form).

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b,\ c \le y \le d$, then..

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

(Apply the fundamental Theorem)

Example: Double Integrals

Calculate $\iint_R f(x,y)\, dA$ for $f(x,y) = 100 - 6x^2y$ and $R: 0 \le x \le 2,\ -1 \le y \le 1$

$= \int_{-1}^1 \int_0^2 (100 - 6x^2y)\, dx\, dy$

$= \int_{-1}^1 (100x - 2x^3y)\Big|_0^2\, dy$

$= \int_{-1}^1 (200 - 16y)\, dy$

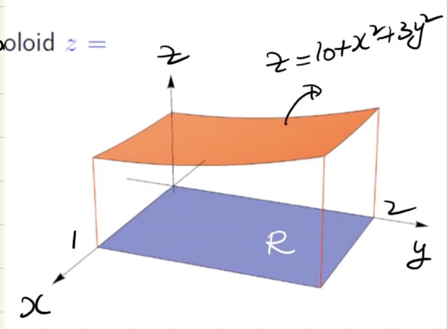
$\boxed{= 400}$

$= \int_0^2 \int_{-1}^1 (100 - 6x^2y)\, dy\, dx$

$\boxed{= 400}$

Example: Double Integrals

Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \le x \le 1,\ 0 \le y \le 2$



Volume $= \iint_R f(x,y)\, dA$

$= \iint_R (10 + x^2 + 3y^2)\, dA$

$= \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$

$= \int_0^1 (10y + x^2y + y^3)\, dx$

$= \int_0^1 (20 + 2x^2 + 8)\, dx$

$\boxed{= 86/3}$

Same value for

$\int_0^2 \int_0^1 (10 + x^2 + 3y^2)\, dx\, dy$

Quiz on 15.1

3/3 100%

~~1.~~ $f(x,y) = x^2 y \qquad [0,2] \times [0,2]$

4 equally sized squares (bottom right)!

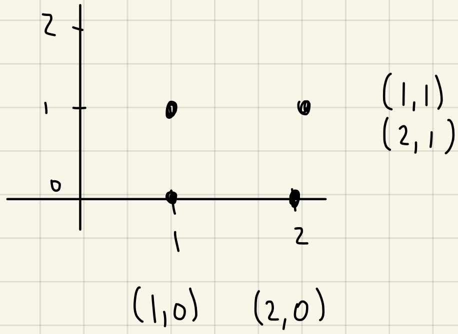
Riemann Sum Approximation!

$[f(1,0) + f(2,0) + f(1,1) + f(2,1)]\,2^2$

$[0 + 0 + 1 + 4]\,4$

$\boxed{= 20}$

$\rightarrow \Delta x \Delta y = 1 \rightarrow \cancel{\boxed{4}}$ $\boxed{5}$



2. $\iint_R \pi y \cos(xy)\, dA$ (Evaluate)

$R: [0,\pi] \times [0,1]$

$a, b \quad c, d$

$x \quad y$

$\boxed{= 2}$

$\int_0^1 \int_0^\pi (\pi y \cos(xy))\, dx\, dy$

3. Volume

$z = 2 - x - y$

$R: [0,1] \times [0,1]$

$\int_0^1 \int_0^1 (2 - x - y)\, dx\, dy \quad \boxed{= 1}$

15.2 & 15.3 — Non-Rectangular Regions

- ☐ Precisely state Fubini's theorem for non-rectangular regions.
- ☐ Write down properties of double integration (that mimic the same properties for single-variable integrals).

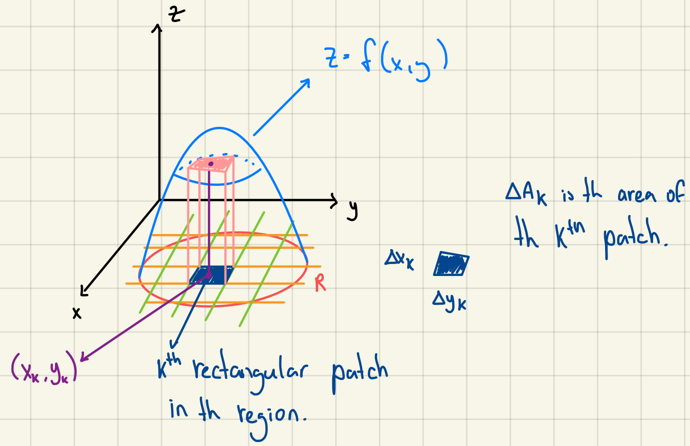
Results for Double Integrals over Non-Rectangular Regions

The Riemann Sum is then given by...

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \cdot \Delta A_k$$

Then in the limit

$$\boxed{\iint_R f(x,y)\, dA = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k}$$

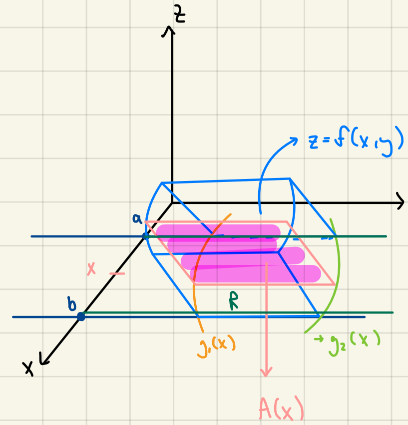


Theorem: Fubini's Theorem (Stronger form)

Let $f(x,y)$ be continuous on a bounded region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$ then

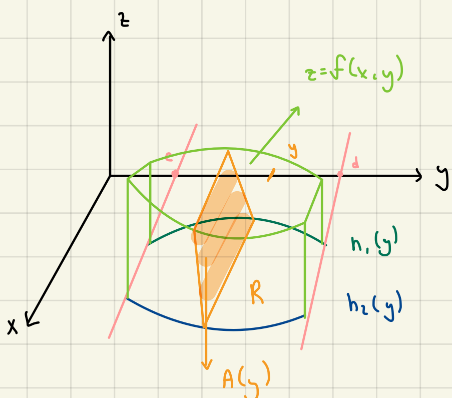
$$\iint_R f(x,y)\, dA = \int_a^b \boxed{\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy}\, dx$$



$$\int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[c,d]$ then

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$



$$A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx$$

Properties of Double Integrals

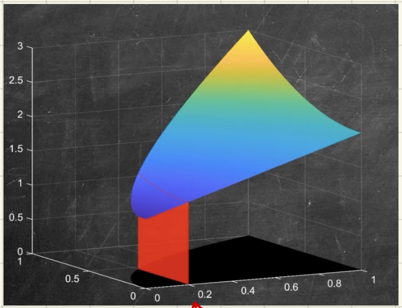
If $f(x,y)$ and $g(x,y)$ are continuous on a bounded region R, then:

1. Constant Multiple: $\int_R\int c\cdot f(x,y)\,dA = c\int_R\int f(x,y)\,dA$, c is constant
2. Sum of Difference: $\int_R\int [f(x,y) \pm g(x,y)]\,dA = \int_R\int f(x,y)\,dA \pm \int_R\int g(x,y)\,dA$
3. $\int_R\int f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R
4. $\int_R\int f(x,y)\,dA \geq \int_R\int g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R
5. If R is the union of two non-overlapping regions R_1 and R_2, then...

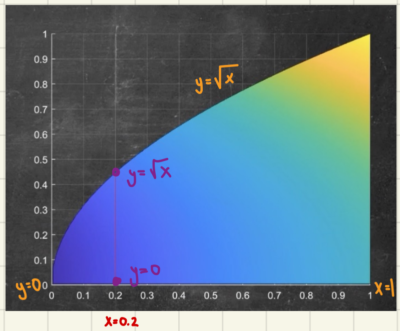
$$\int_R\int f(x,y)\,dA = \int_{R_1}\int f(x,y)\,dA + \int_{R_2}\int f(x,y)\,dA \qquad R = R_1 \cup R_2$$

☐ Integrate a function over a nonrectangular region using two orders of integration.

Double Integrals over NonRectangular Regions



A(x) $\quad$ x=0.2 $\quad$ A(0.2)



Ex. Compute Volume under $f(x,y) = 1+x+y^2$
Bounded by x-axis, $x=1$ and $y=\sqrt{x}$

$$V = \int_0^1 A(x)\,dx = \int_0^1\int_0^{\sqrt{x}} 1+x+y^2\,dy\,dx$$

$$= \int_0^1 y + xy + y^3/3 \Big|_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \sqrt{x} + x^{3/2} + x^{3/2}/3 \; dx$$

$$= 2/3\, x^{3/2} + 4/3 \cdot 2/5\, x^{5/2} \Big|_0^1$$

$$= 2/3 + 8/15 = 18/15 = \boxed{6/5}$$

Ex.

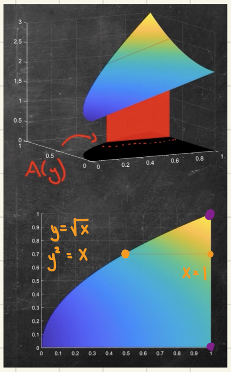
$$V = \int_0^1 A(y)\,dy = \int_0^1\int_{y^2}^1 1+x+y^2\,dx\,dy$$

$$= \int_0^1 x + x^2/2 + y^2 x \Big|_{y^2}^1 dy$$

$$= \int_0^1 (3/2 + \cancel{y^2}) - (\cancel{y^2} + y^4/2 + y^4)\,dy$$

$$= 3/2\, y - 3/2\, y^5/5 \Big|_0^1$$

$$= 3/2 - 3/10 = 15-3/10 = \boxed{6/5}$$



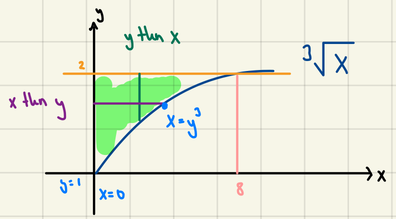
☐ Integrate a function over a non-rectangular region more easily by changing the order of integration.

Changing Order of Integration

$$\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\, dy\, dx \qquad \int_0^2\int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy = \int_0^2 \frac{y^3}{y^4+1}\, dy$$

$$= \tfrac{1}{4} \ln|y^4+1| \Big|_0^2$$

$$= (\ln 17)/4$$



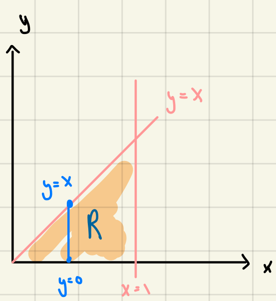
☐ Examples on Integration over a non-rectangular region

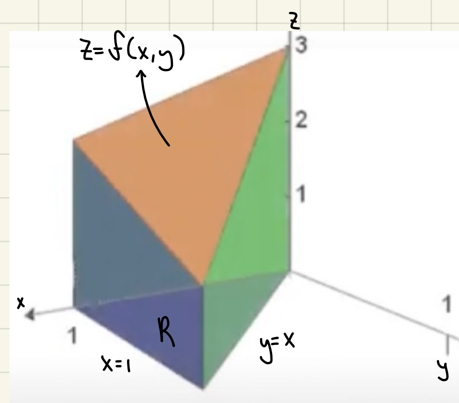
Double Integrals Over Non-Rectangular Regions

Ex.

Find the volume of the prism whose base is the triangle in the xy−plane bounded by the x−axis and the lines $y=x$ and $x=1$ and whose top lies in the plane

$$z = 3 - x - y$$





$$\text{Volume} = \iint_R (3-x-y)\, dA$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x} (3-x-y)\, dy\, dx$$

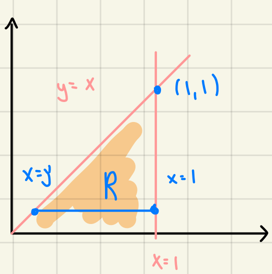
$$= \int_0^1 \left(3y - xy - y^2/2\right)\Big|_0^x dx$$

$$= \int_0^1 \left(3x - 3x^2/2\right) dx \qquad \boxed{= 1}$$

Ex.

Find the volume of the prism whose base is the triangle in the xy−plane bounded by the x−axis and the lines $y=x$ and $x=1$ and whose top lies in the plane

$$z = 3 - x - y$$



$$\text{Volume} = \iint_R (3-x-y)\, dA$$

$$= \int_{y=0}^{y=1}\int_{x=y}^{x=1} (3-x-y)\, dx\, dy$$

$$= \int_0^1 \left(3x - x^2/2 - xy\right)\Big|_y^1 dy$$

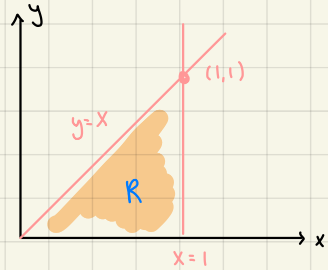
$$= \int_0^1 \left(5/2 - 4y + 3/2\, y^2\right) dy \qquad \boxed{= 1}$$

Ex.

Calculate

$$\iint_R \frac{\sin x}{x}\, dA$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.



$$\int_R\int \frac{\sin x}{x}\, dA = \int_0^1\int_{x=y}^{x=1} \frac{\sin x}{x}\, dx\, dy$$

$$\int_R\int \frac{\sin x}{x}\, dA = \int_0^1\int_0^x \left(\frac{\sin x}{x}\right) dy\, dx$$

$$= \int_0^1 \left(\frac{\sin x}{x} \cdot y\right)\Big|_0^x dx$$

$$= \int_0^1 \sin x\, dx$$

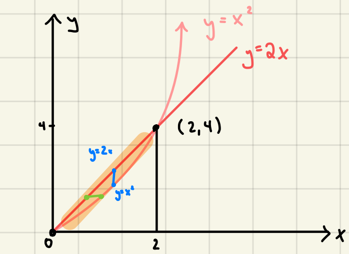
$$= \Big[-\cos x\Big]_0^1 \qquad \boxed{= -\cos(1) + 1 \approx 0.5}$$

Ex.

Sketch the region of integration for the integral

$$\int_0^2\int_{x^2}^{2x} (4x+2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.



$y = x^2$ → $x = \sqrt{y}$ $\qquad$ $y = 2x$ → $x = y/2$

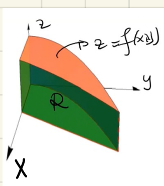
$y = x^2$ to $y = 2x$

$x = 0$ to $x = 2$

$$\boxed{\int_{y=0}^{y=4}\int_{x=y/2}^{x=\sqrt{y}} (4x+2)\, dx\, dy}$$

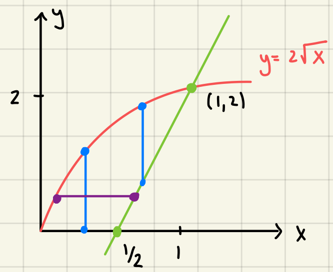
Ex.

Find the volume of the wedgelike solid that lies beneath the sruface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



$2\sqrt{x} = 4x - 2$

$\sqrt{x} = 2x - 1$



$y = 4x - 2$

$x = (y+2)/4$

$y = 2\sqrt{x}$

$x = y^2/4$

* → 2 seperate integrals needed

$$\text{Volume} = \int_R\int (16 - x^2 - y^2)\, dA$$

$$= \int_{y=0}^{y=2}\int_{x=y^2/4}^{x=(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left(16x - x^3/3 - xy^2\right)\Big|_{y^2/4}^{\frac{y+2}{4}} dy$$

$$= \int_0^2 \left[4(y+2) - (y+2)^3/3\cdot 64 - (y+2)y^2/4 - 4y^2 + y^6/192 + y^4/4\right] dy$$

⋮

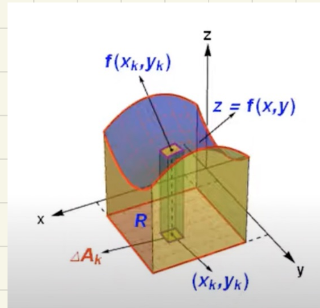
$$\boxed{\approx 12.4}$$

Examples on computing area using double integration

Area Using Double Integration

We then created the Riemann sums as:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$



How about the case when $f(x,y)=1$.

The Riemann sums takes the form

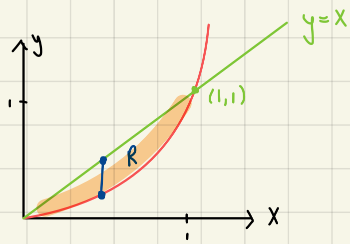
$$S_n = \sum_{k=1}^{n} \Delta A_k$$

Now as the norm of the partition of R approaches zero, we define the area of R to be the limit

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA$$

Ex.

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



$$\text{Area} = \iint_R dA$$

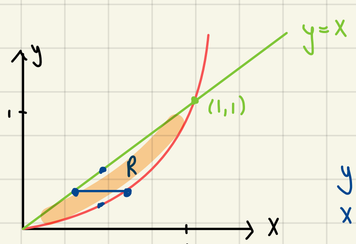
$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} (1)\,dy\,dx$$

$$= \int_0^1 [y]_{x^2}^{x}\,dx$$

$$= \int_0^1 (x-x^2)\,dx \;\boxed{= 1/6}$$

Ex.

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



$y = x^2$
$x = \sqrt{y}$

$$\text{Area} = \iint_R (1)\,dx\,dy$$

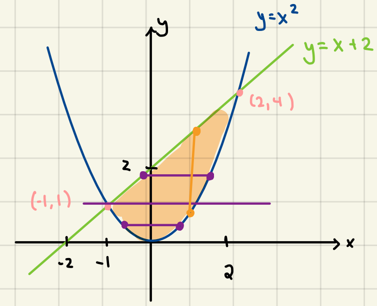
$$= \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} (1)\,dx\,dy$$

$$= \int_0^1 [x]_y^{\sqrt{y}}\,dy$$

$$= \int_0^1 (\sqrt{y} - y)\,dy = \left(2y^{3/2}/3 - y^2/2\right)_0^1 = 2/3 - 1/2 \;\boxed{= 1/6}$$

Ex.

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.



$$\text{Area} = \iint_R (1)\, dA$$

$$x \longrightarrow \quad = \int_{y=0}^{y=1}\int_{x=-\sqrt{y}}^{x=\sqrt{y}} (1)\,dx\,dy + \int_{y=1}^{y=4}\int_{x=y-2}^{x=\sqrt{y}} (1)\,dx\,dy$$

$$y \longrightarrow \quad = \int_{x=-1}^{x=2}\int_{y=x^2}^{y=x+2} (1)\,dy\,dx \quad \longleftarrow \text{Winner!}$$

$$= \int_{-1}^{2} \left[(x+2) - x^2\right] dx$$

$$= 9/2$$

$$x + 2 = x^2$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \; , \; x = 2$$

Ex.

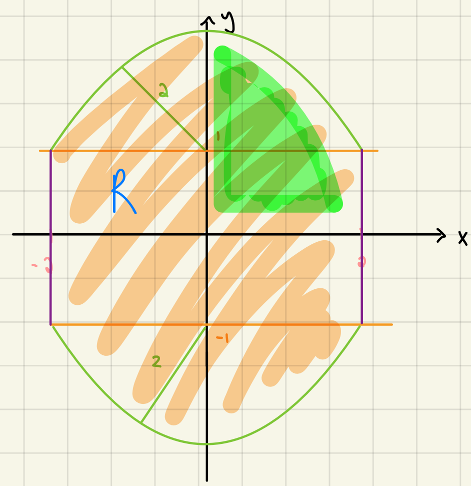
Find the area of the playing field described by

$$R: \qquad -2 \le x \le 2, \quad -1 - \sqrt{4 - x^2} \le y \le 1 + \sqrt{4 - x^2}$$

using:

(a) Fubini's Theorem

(b) Simple geometry



a) $$A = \iint_R (1)\, dA$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=1+\sqrt{4-x^2}} (1)\, dA$$

$$= \int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx$$

$$= (2 + \pi)$$

Area of th region

$$R = 4(2 + \pi)$$
$$= 8 + 4\pi$$

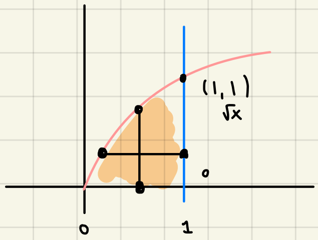
b) Rectangle $= 4 \times 2$
$= 8$

Circular Part $= \pi 2^2$
$= (4\pi)$

Quiz on 15.2 – 15.3

100%

1.



$y=\sqrt{x}$

$x=y^2$

$$V = \iint_R f(x,y)\,dA$$

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=1} x \, dx \, dy$$

$$x^2/2$$

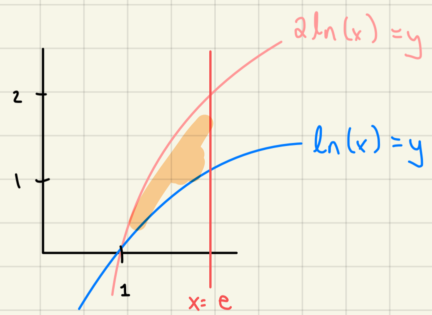
$$\int_0^1 1-y^4/2 \; dy$$

$$= 2/5 \quad = 0.4$$

2. $y = \ln(x)$
$y = 2\ln(x)$
$x = e$

1st quadrant

Area



$$A = \iint_R (1)\, dA$$

$$= \int_1^e \int_{\ln(x)}^{2\ln(x)} (1)\, dy\, dx$$

$$= 1$$

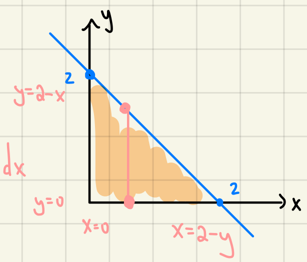
3. Volume $z = 12-3y^2$
$x+y=2$

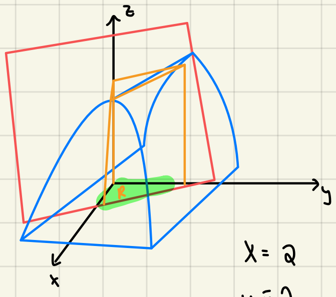
first octant

$$\iint_R (12-3y^2)\, dA$$

$$\int_{x=0}^{x=2} \int_{y=0}^{y=2-x} (12-3y^2)\, dy\, dx$$

$$= 20$$





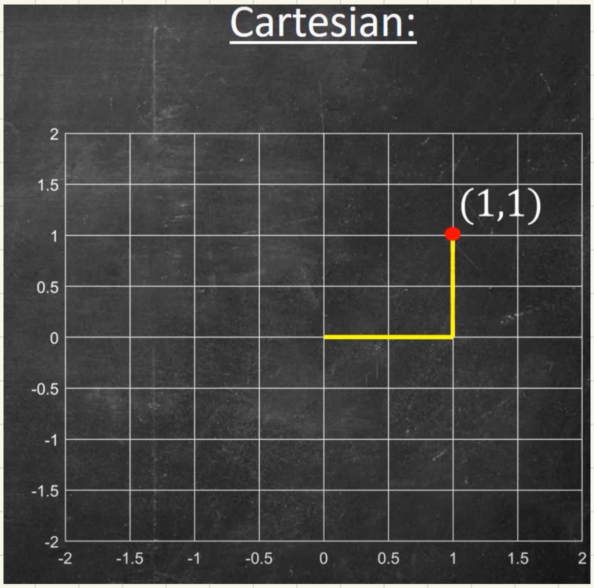
$x = 2$

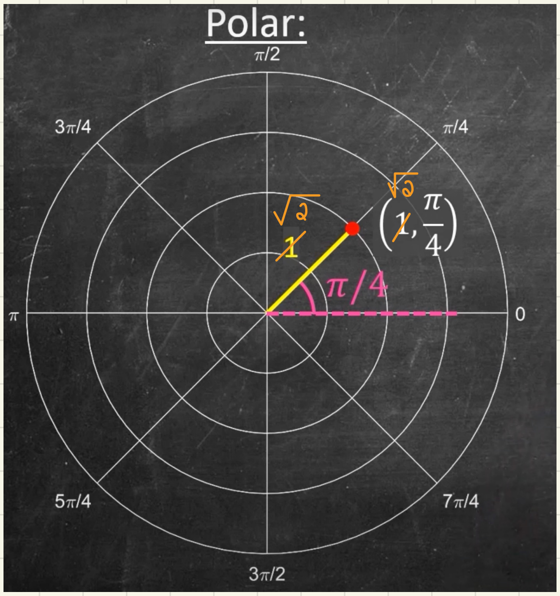
$y = 2$

15.4 — Polar Double Integrals

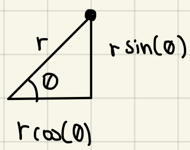
- ☐ Use polar coordinates to describe regions in the plane (review from section 11.5).
- ☐ Write down and utilize the formula for double integrals using polar coordinates.

Intro to Double Integrals using Polar Coordinates



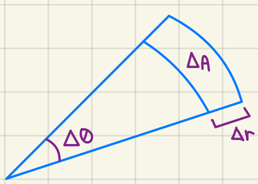


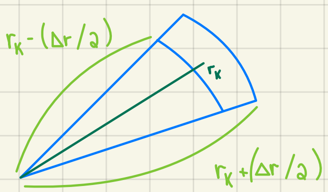
Note:



$$\boxed{\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}}$$

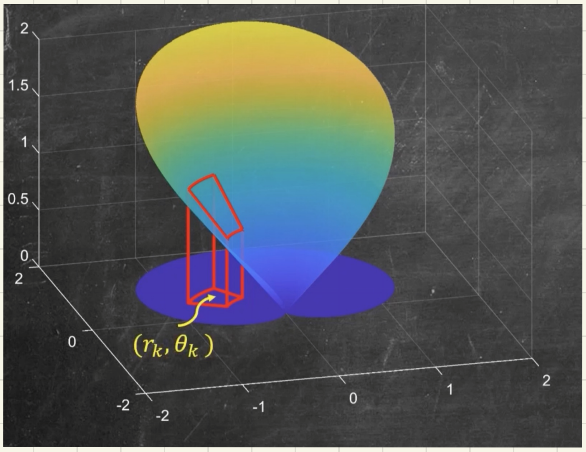
ΔA = Area big wedge − Area small wedge





$$= \frac{\Delta\theta}{2\pi}\,\pi\left(\left[r_k + \frac{\Delta r}{2}\right]^2 - \left[r_k - \frac{\Delta r}{2}\right]^2\right)$$

$$\boxed{= r_k\,\Delta r\,\Delta\theta}$$



$$\Delta V \approx f(r_K, \theta_K)\, \Delta A$$
$$= f(r_K, \theta_K)\, r_K\, \Delta r\, \Delta \theta$$

$$V \approx \sum_{K=1}^{n} f(r_K, \theta_K)\, r_K\, \Delta r\, \Delta \theta$$

$$\boxed{= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta}$$

Ex. $f(r,\theta) = r$ above cardioid $r = 1 - \sin(\theta)$

$$V = \int_0^{2\pi} \int_0^{1-\sin(\theta)} r \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} r^3/3 \Big|_0^{1-\sin\theta} d\theta$$

$$= \int_0^{2\pi} (1-\sin(\theta))^3/3\, d\theta$$

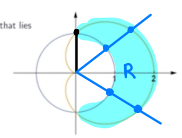
$$= 5\pi/3$$

- [ ] Apply the polar coordinates double integral formula to compute areas and volumes.

Examples of Polar Coordinates Double Integrals

Example-1: Double integrals

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.



$r = 1 + \cos(\theta)$, $r = 1$

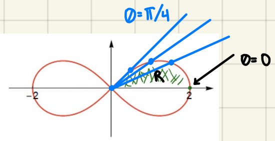
$1 = 1 + \cos(\theta)$

$\Rightarrow \cos(\theta) = 0, \quad \theta = \pm \pi/2$

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=1}^{r=1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$

Example-2: Double integrals

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.



$$\text{Area} = \iint_R (1)\, r\, dr\, d\theta$$

We have $r^2 = 4\cos(2\theta)$, when $\theta = 0$, $r = 2$

when $\theta = \pi/4$, $r^2 = 4\cos(\pi/4)$

$\Rightarrow r = 0$

θ goes from 0 to $\pi/4$,

r goes from 0 to $\sqrt{4\cos(2\theta)}$

$$\Rightarrow \text{Area} = \int_0^{\pi/4} \int_0^{\sqrt{4\cos(2\theta)}} (1)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} r^2/r \Big|_0^{\sqrt{4\cos(2\theta)}} d\theta$$

$$= \int_0^{\pi/4} 2\cos(2\theta)\, d\theta$$

$$= \cancel{2}\left[\sin(2\theta)/\cancel{2}\right]_0^{\pi/4}$$

$$= 1$$

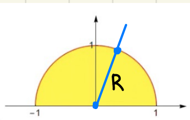
$$\boxed{\text{Required area} = 4(1) = 4.}$$

Example-3: Double integrals

Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx$$

where R is the semicircular region bounded by the x−axis and the curve $y = \sqrt{1-x^2}$.



$x = r\cos\theta$, $y = r\sin\theta$

$\rightarrow e^{x^2+y^2} = e^{r^2}$

$\rightarrow dy\, dx = r\, dr\, d\theta$

$$\iint_R e^{x^2+y^2}\, dy\, dx$$

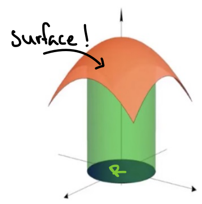
$$= \frac{1}{2}\int_0^{\pi} \int_{r=0}^{r=1} \underline{e^{r^2}(2r)} dr\, d\theta$$

$$= \pi/2 \int_0^1 e^{r^2}(2r)\, dr$$

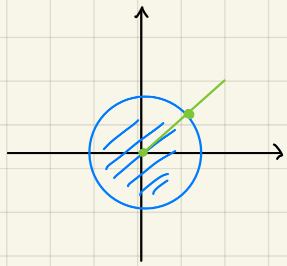
$$= \pi/2\,(e-1)$$

Example-5: Double integrals

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy−plane.



$$\text{Volume} = \int_R\int (9 - x^2 - y^2)\, dx\, dy$$



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\text{Volume} = \int_R\int 9 - (x^2 + y^2)\, dx\, dy$$

$$= \int_R\int (9 - r^2)\, r\, dr\, d\theta$$

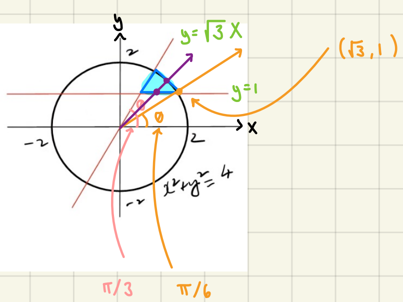
$$= \int_0^{2\pi}\int_{r=0}^{r=1} (9 - r^2)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^1 (9 - r^2)\, r\, dr$$

$$\boxed{= 17\pi/2}$$

Example-6: Double integrals

Using polar integration, find the area of the region R in the xy−plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$ and below the line $y = \sqrt{3}x$.



$$\text{Area} = \int_R\int (1)\, r\, dr\, d\theta$$

$x^2 + y^2 = 4$ when $y = 1$

$x^2 + 1 = 4 \longrightarrow x = \sqrt{3}$

Slope of the orange line: $m = 1/\sqrt{3} \longrightarrow \tan\theta = 1/\sqrt{3} \longrightarrow \theta = \pi/6$

The line $y = \sqrt{3}x$ has a slope of $\sqrt{3}$. $\longrightarrow \tan\theta = \sqrt{3} \longrightarrow \theta = \pi/3$

r ends at the circle ; $r = 2$

For the lower limit: $y = 1 \longrightarrow r\sin\theta = 1$

$\longrightarrow r = 1/\sin\theta = \csc\theta$

$x = r\cos\theta$
$y = r\sin\theta$

r goes from $r = 1$ to $r = \csc\theta$

$$\text{Area} = \int_{\theta=\pi/6}^{\theta=\pi/3} \int_{r=\csc\theta}^{r=2} (1)\, r\, dr\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \left[r^2/2 \right]_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2} \int_{\pi/6}^{\pi/3} (4 - \csc^2\theta)\, d\theta$$

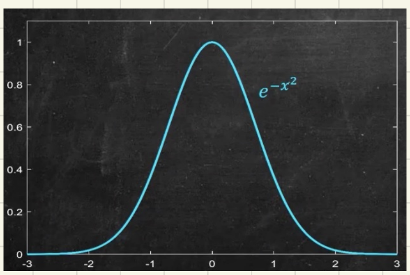
$$= (\pi - \sqrt{3})/3$$

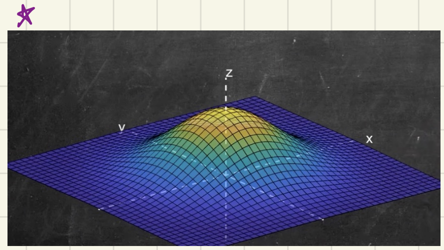
☐ Evaluate a single-variable integral by relating it to a double integral (in polar coordinates!).

An Interesting Example — The Gaussian Integral

Gaussian Integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$





$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

convert to polar!

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$u = -r^2$
$du = -2r\,dr$

$u(0) = 0$
$u(\infty) = -\infty$

$$= \int_{0}^{2\pi} \int_{0}^{-\infty} -\frac{1}{2} e^{u}\, du\, d\theta$$

$$= \int_{0}^{2\pi} \left[-\frac{1}{2} e^{u} \Big|_{0}^{-\infty} \right] d\theta$$

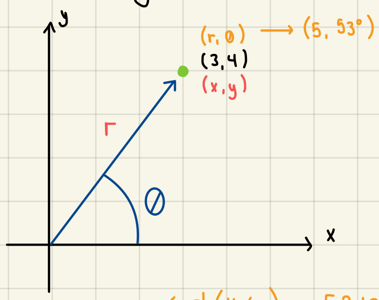
$$= \int_{0}^{2\pi} \left[0 + 1/2 \right] d\theta$$

$$= 2\pi \cdot \frac{1}{2} = \pi$$

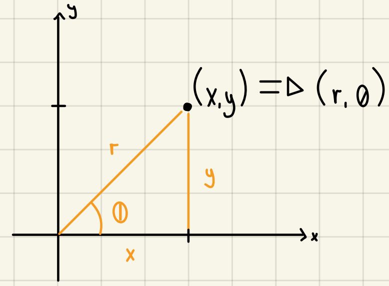
$$I^2 = \left(\int_{\infty}^{\infty} e^{-x^2}\, dx \right)^2 = \pi$$

$$I = \int_{\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Khan Academy — Polar Coordinates



$\tan^{-1}(4/3) = 53.13°$



- △ $r^2 = x^2 + y^2$
- △ $\tan\theta = y/x$

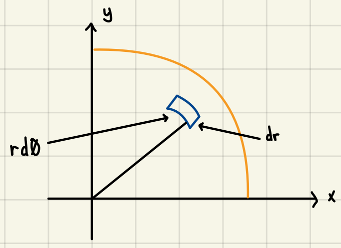
  $\theta = \tan^{-1}(y/x)$
- △ $\sin\theta = y/r$

  $y = r\sin\theta$
- △ $\cos\theta = x/r$

  $x = r\cos\theta$

Double Integrals in Polar Coordinates

$$\iint_R f(x,y)\,dA \longrightarrow \iint_R f(r,\theta)\, r\, dr\, d\theta$$



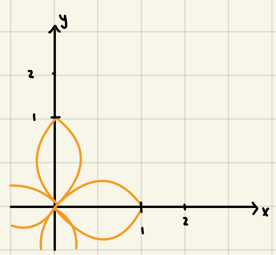
To turn radians (since they are not a unit of length) into a bit of arc length, we must mulitiply by r. Thus ...

$$dA = (r\,d\theta)(dr)$$

Example: Evaluate this double integral!

$$\int_0^2\int_0^{2\pi} r^3\, d\theta\, dr = 8\pi$$

Example:



$f(r,\theta) = r\sin(\theta)$

Let R be flower-shaped region, defined by $r \leq \cos(2\theta)$

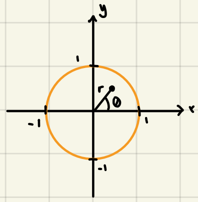
$$\iint_R r^2\sin(\theta)\,dr\,d\theta \longrightarrow \int_0^{2\pi}\int_0^{\cos(2\theta)} r^2 \sin(\theta)\,dr\,d\theta = 0$$

Triple Integrals in Cylindrical Coordinates

- [ ] $dV = r\,d\theta\,dr\,dz$

$$\iiint_R f(r,\theta,z)\,dV = \iiint_R f(r,\theta,z)\, r\,d\theta\,dr\,dz$$

Example: Volume of a Sphere



$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$
$$-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$$

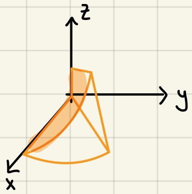
In cylindrical coordinates, the distance from a point (r, θ, z) to the origin is $\sqrt{r^2+z^2}$

$\rightarrow r^2 = x^2+y^2$; $\sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$

$\rightarrow$ The unit sphere $r^2+z^2 = 1^2 \rightarrow z = \pm\sqrt{1-r^2}$

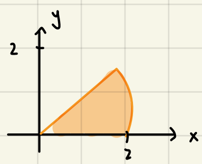
$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r \, dz\,dr\,d\theta = 4\pi/3$$

Example: Integrating Over a Pie Slice



$$x \ge 0,\ y \ge 0,\ z \ge 0,\ y \le x$$
$$x^2+y^2 \le 4,\ z \le y/x$$

$$f(x,y,z) = z^2 - x^2 - y^2 \longrightarrow f(r,\theta,z) = z - r^2$$



$x \ge 0,\ y \ge 0,\ y \le x \longrightarrow 0 \le \theta \le \pi/4$

$x^2+y^2 \le 4 \qquad 0 \le r \le r$

$z \le y/x \longrightarrow \tan(\theta) = y/x \longrightarrow 0 \le z \le \tan(\theta)$

$$\int_0^{\pi/4}\int_0^2\int_0^{\tan(\theta)} f \, dV$$

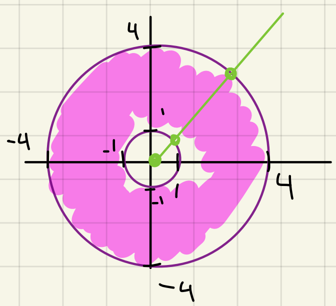
$$\therefore \int_0^{\pi/4}\int_0^2\int_0^{\tan(\theta)} (z-r^2) r \, dz\,dr\,d\theta =$$

Quiz on 15.4

100%

1. Area of R

$R = \{(x,y) \mid 1 \le x^2 + y^2 \le 16\}$.



$A = r_k \Delta r \Delta \theta$

$= \int_R \int (1)\, r\,dr\,d\theta$

$= \int_0^{2\pi} \int_1^{16} r\,dr\,d\theta \longrightarrow \int_1^4$

$= \int_0^{2\pi} \frac{r^2}{2} \Big|_1^{16} d\theta \qquad = 15\pi$ ✓

$\frac{16^2}{2} - 1^2/2$ ¡ 4 Not! 16 !

$= 2\pi\,(127.5)$

$\boxed{= 255\pi}$ ✗

2. Evaluate the double integral

$\int_0^{\pi/2} \int_1^{1+\cos\theta} \left((8r)/(8+\pi) \right) dr\,d\theta$

$\boxed{= 1}$ ✓

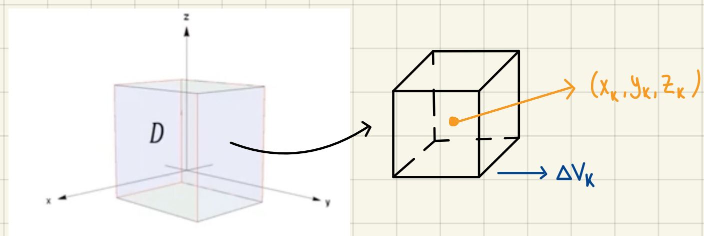
15.5 — Triple Integrals

☐ Explain the construction of a triple integral in rectangular coordinates using Riemann Sums.

Construction of Triple Integrals

We then create the Riemann sums as:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$



Now as the norm of the partition D approaches zero:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k = \iiint_D f(x,y,z)\,dV$$

How about the case when $f(x,y,z) = 1$?

The Riemann sums take the form

$$S_n = \sum_{k=1}^{n} \Delta V_k$$

Again, in the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV$$

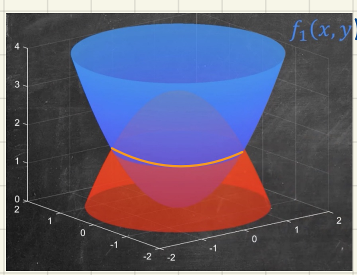
Which gives us the Volume of the closed, bounded region D in space.

☐ Compute the volume of a region using a triple integral.

Volume Between Two Surfaces Via Triple Integral

$$x^2+y^2 = 3-x^2-y^2$$
$$= x^2+y^2 = 3/2$$

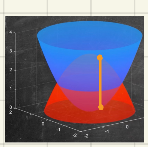
Circle of intersection!



$f_1(x,y) = x^2+y^2$

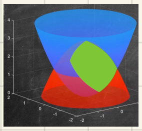
$f_2(x,y) = 3-x^2-y^2$

$$\boxed{V = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} dz\,dy\,dx}$$

Ex. 

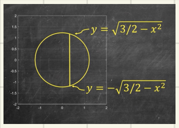
$$V = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=x^2+y^2}^{z=3-x^2-y^2} dz\,dy\,dx$$

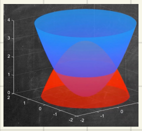
↓



$$V = \int_{x=a}^{x=b} \int_{y=-\sqrt{3/2 - x^2}}^{y=\sqrt{3/2-x^2}} \int_{z=x^2+y^2}^{z=3-x^2-y^2} dz\, dy\, dx$$

↓





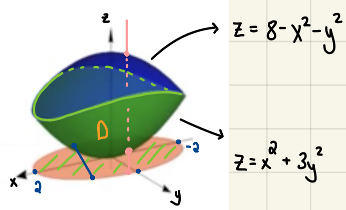
$$V = \int_{x=-\sqrt{3/2}}^{x=\sqrt{3/2}} \int_{y=-\sqrt{3/2 - x^2}}^{y=\sqrt{3/2-x^2}} \int_{z=x^2+y^2}^{z=3-x^2-y^2} dz\, dy\, dx$$

- ☐ Set up and compute Triple Integrals in Rectangular Coordinates.
- ☐ Compute the average value of a function over a solid (3D) region.

Examples!

Example-4: Triple integrals

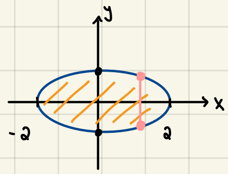
Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



$$\text{Volume} = \iiint_D (1)\, dz\, dy\, dx$$

The z-variable goes from $z = x^2 + 3y^2$ to $z = 8 - x^2 - y^2$

$$x^2 + 3y^2 = 8 - x^2 - y^2$$
$$2x^2 + 4y^2 = 8$$
$$\rightarrow x^2 + 2y^2 = 4$$
$$\rightarrow 2y^2 = 4 - x^2$$
$$\rightarrow y = \pm\sqrt{(4-x^2)/2}$$



The y-variable goes from $y = -\sqrt{(4-x^2)/2}$ to $y = \sqrt{(4-x^2)/2}$

The x-variable goes from $x = -2$ to $x = 2$.

$$\text{Volume} = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} (1)\, dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8 - 2x^2 - 4y^2\right) dy\, dx$$

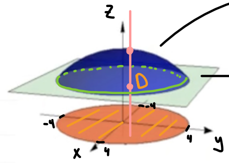
$$= \int_{-2}^{2} \Big[(8 - 2x^2)y - 4y^3/3 \Big]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx$$

$$\vdots$$

$$\boxed{= 8\pi\sqrt{2}}$$

Example-1: Triple integrals

Let S be the sphere of radius 5 centered at the origin, and let D be the region under the sphere that lies above the plane $z=3$. Set up the limits of integration for evaluating the triple integral of a function $f(x,y,z)$ over the region D.



$x^2+y^2+z^2=25$

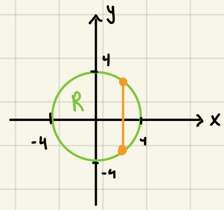
$z=3$

We have $x^2+y^2+z^2=25$

$$\iiint_D f(x,y,z)\,dV = \iiint_D f(x,y,z)\,dz\,dy\,dx$$

For z-variable, we go from $z=3$ to $z=\sqrt{25-x^2-y^2}$

$$\iiint_{z=3}^{z=\sqrt{25-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$



$x^2+y^2+z^2=25$ with $z=3$

$\to\ x^2+y^2+9+25$

$\to\ x^2+y^2=16$

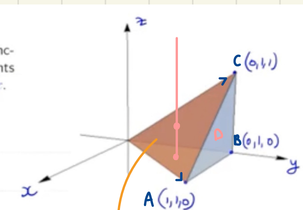
For y-variable, we go from $y=-\sqrt{16-x^2}$ to $y=\sqrt{16-x^2}$

$$\iint_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}\int_{z=3}^{z=\sqrt{25-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

$$\boxed{\int_{x=-4}^{x=4}\int_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}\int_{z=3}^{z=\sqrt{25-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx}$$

Example-2: Triple integrals

Set up the limits of integration for evaluating the triple integral of a function $f(x,y,z)$ over the tetrahedron D whose vertices are given by the points $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$. Use the order of integration $dz\,dy\,dx$.



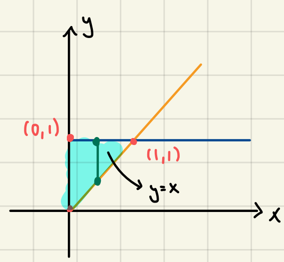
Equation of this plane is $z=y-x$

The required integral is ...

$$\iiint_D f(x,y,z)\,dv = \boxed{\int_{x=0}^{x=1}\int_{y=x}^{y=1}\int_{z=0}^{z=y-x} f(x,y,z)\,dz\,dy\,dx}$$

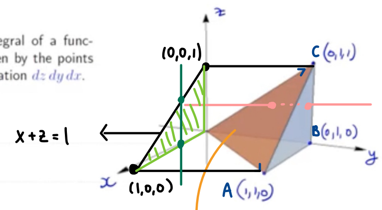
For our z-variable, we go from $z=0$ to $z=y-x$.

For the y-variable, we go from $y=x$ up to $y=1$



Example-2: Triple integrals

Set up the limits of integration for evaluating the triple integral of a function $f(x, y, z)$ over the tetrahedron D whose vertices are given by the points $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$. Use the order of integration $dz\,dy\,dx$.



Equation of this plane is $z = y - x$

$$V = \int_{x=0}^{x=1} \int_{z=0}^{z=1-x} \int_{y=z+x}^{y=1} (1)\, dy\, dz\, dx$$

$$= \int_{x=0}^{x=1} \int_{z=0}^{z=1-x} \Big[y \Big]_{z+x}^{1} dz\, dx$$

$$= \int_{x=0}^{x=1} \int_{z=0}^{z=1-x} (1 - x - z)\, dz\, dx$$

$$= \int_{x=0}^{x=1} \Big[(1-x)z - z^2/2 \Big]_0^{1-x} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=1} (1 - x^2)\, dx$$

$$\boxed{= 1/6}$$

Summary

- For $y = f(x)$ over an interval $[a, b]$

$$\text{Average value} = \underbrace{\frac{1}{b-a}}_{\text{length of } [a,b]} \int_a^b f(x)\, dx$$

- For $f(x,y)$ defined on a region R is the plane

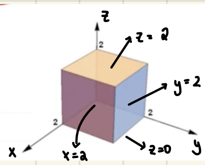
$$\text{Average value} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

- For $f(x,y,z)$ defined on a region D in space

$$\text{Average value} = \frac{1}{\text{Volume}(D)} \iiint_D f(x,y,z)\, dV$$

Example-5: Triple integrals

Find the average value of $f(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2, y = 2$ and $z = 2$ in the first octant.



$$\text{Volume} = 2 \times 2 \times 2 = 8$$

$$\text{Average value of } f(x,y,z) = \frac{1}{\text{Vol}(D)} \iiint_D (xyz)\, dV.$$

$$= \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 (xyz)\, dx\, dy\, dz$$

$$= \frac{1}{8} \int_0^2 \int_0^2 \frac{x^2 yz}{2} \Big|_0^2 \, dy\, dz$$

$$= \frac{1}{4} \int_0^2 \int_0^2 (yz)\, dy\, dz$$

$$= \frac{1}{2} \int_0^2 (z)\, dz$$

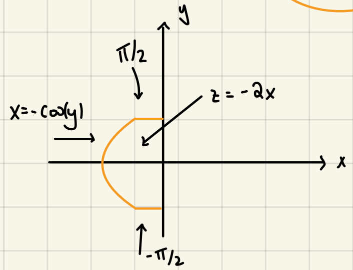
$$\boxed{= 1}$$

Average value of $f(x,y,z)$ over the region D is 1.

Quiz on 15.5

100%

1.



$$\int_{-\pi/2}^{\pi/2} \int_{-\cos(y)}^{0} \int_{0}^{-2x} (1) \; dz \, dx \, dy$$

$$\boxed{= \pi/2 \approx 1.57}$$

2. $z = 16 - 2(x^2 + y^2)$ $\qquad f(x,y,z) = 15/\pi \sqrt{x^2+y^2}$

$z = 2(x^2+y^2)$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=16-2(x^2+y^2)}^{z=2(x^2+y^2)} 15/\pi \sqrt{x^2+y^2} \; dV$$

$$16 - 2(x^2+y^2) = 2(x^2+y^2)$$

$$16 = 4(x^2+y^2)$$

$$4 = (x^2+y^2)$$

$$y = \pm \sqrt{4-x^2}$$

$$x = \pm 2$$

$$\boxed{= -512}$$

15.6 — Moments and Center of Mass

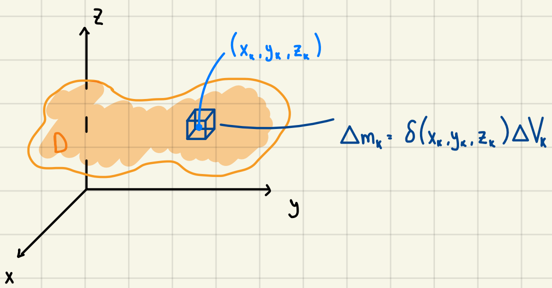
This section shows how to calculate the masses and moments of two- and three-dimensional objects in Cartesian coordinates. We also look at how multiple integrals are used to compute probabilities.

Masses and First Moments

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\,\Delta V_k$$

$$\boxed{= \iiint_D \delta(x,y,z)\,dV}$$



Mass and First Moment Formulas

Three-Dimensional Solid

- Mass: $M = \iiint_D \delta\, dV$, $\delta = \delta(x,y,z)$ is the density at (x,y,z).
- First Moments About the Coordinate Planes:

$$M_{yz} = \iiint_D x\delta\, dV, \quad M_{xz} = \iiint_D y\delta\, dV, \quad M_{xy} = \iiint_D z\delta\, dV.$$

- Center of Mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

Two-Dimensional Plate

- Mass: $M = \iint_R \delta\, dA$, $\delta = \delta(x,y)$ is the density at (x,y).
- First Moments:

$$M_y = \iint_R x\delta\, dA, \quad M_x = \iint_R y\delta\, dA$$

- Center of Mass:

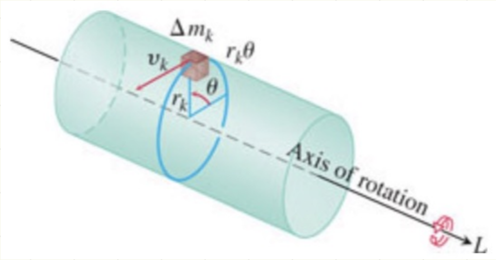
$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

Moments of Inertia

If the shaft rotates at a constant angular velocity of $\omega = d\theta/dt$ radians per second, the block's center of mass will trace its orbit at a linear speed of...

$$\boxed{v_k = \frac{d}{dt}(r_k\theta) = r_k\frac{d\theta}{dt} = r_k\omega.}$$

Where r_k denotes the distance from the kth block's center of mass to the axis of rotation.



The block's kinetic energy will be approximately...

$$\tfrac{1}{2}\Delta m_k v_k^2 = \tfrac{1}{2}\Delta m_k (r_k\omega)^2 = \tfrac{1}{2}\omega^2 r_k^2 \Delta m_k.$$

The kinetic energy will be approximately...

$$\sum \tfrac{1}{2}\omega^2 r_k^2 \Delta m_k.$$

The integral approached by these sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy..

$$KE_{shaft} = \int \tfrac{1}{2}\omega^2 r^2\, dm = \tfrac{1}{2}\omega^2 \int r^2\, dm.$$

The factor ...

$$I = \int r^2\, dm$$

is the moment

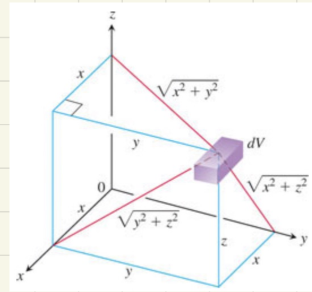
$$KE_{shaft} = \tfrac{1}{2} I \omega^2.$$

The moment of inertia about L of the entire object is...

$$\boxed{\begin{aligned} I_L &= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n\to\infty} \sum_{k=1}^{n} r^2(x_k, y_k, z_k)\,\delta(x_k, y_k, z_k)\,\Delta V_k \\ &= \iiint_D r^2 \delta\, dV. \end{aligned}}$$

If L is the x-axis, then $r^2 = y^2 + z^2$ and

$$I_x = \iiint_D (y^2+z^2)\,\delta(x,y,z)\,dV.$$



And...

$$I_y = \iiint_D (x^2+z^2)\,\delta(x,y,z)\,dV, \quad I_z = \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV.$$

TABLE 15.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID		
About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\,dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\,dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\,dV$	
About a line L:	$I_L = \iiint r^2(x, y, z)\,\delta\,dV$	$r(x, y, z) =$ distance from the poing (x, y, z) to line L
TWO-DIMENSIONAL PLATE		
About the x-axis:	$I_x = \iint y^2\,\delta\,dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\,\delta\,dA$	
About a line L:	$I_L = \iint r^2(x, y)\,\delta\,dA$	$r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\,\delta\,dA = I_x + I_y$	

Probability

Remember: $P(a \leq X \leq b) = \int_a^b f(x)\,dx.$

The probability that a pair of random variables (X, Y) takes values lying within a particular region is determined by a joint probability density function f. Integrating the joint probability density function over a region R in the plane gives the probability that a pair of random variables take values in the region:

$$P((X,Y) \in R) = \iint_R f(x,y)\,dx\,dy.$$

If the region is a rectangle, then this expression has the simple form:

$$P(a \leq X \leq b \text{ and } c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\,dx\,dy.$$

Definition: A joint probability density function f is a function that satisfies three conditions:

1. $f(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
3. $P((X,Y) \in R) = \iint_R f(x,y)\,dx\,dy.$

Note: A pair of random variables has a uniform distribution on a region R with finite area A if $f(x,y) = 1/A$ for any $(x,y) \in R$, and $f(x,y) = 0$ otherwise.

Means and Expected Values

The mean, or expected value, of a random variable was seen to be:

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

When X and Y have joint probability density function f, the expected value of X and the expected value of Y are:

$$\mu_X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy,$$

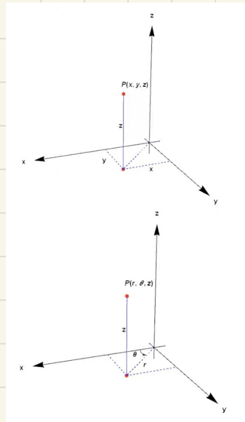
$$\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy.$$

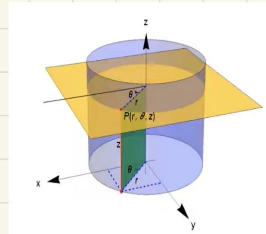
15.7 – Integrals in Cylindrical & Spherical Coordinates (Centroid p. 932–933)

- ☐ Convert from Cartesian to cylindrical coordinates and vice versa
- ☐ Compute triple integrals in cylindrical coordinates

Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates





Cylindrical Coordinates represent a point P in space by order triple (r, θ, z).

Equations that relate Cartesian Coordinates $P(x,y,z)$ with Cylindrical Coordinates $P(r,\theta,z)$ are:

$x = r\cos\theta$ $\qquad r = \sqrt{x^2+y^2}$

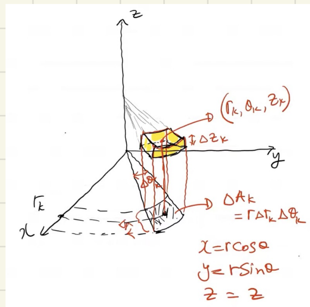
$y = r\sin\theta$ $\qquad \theta = \tan^{-1}(y/x)$

$z = z$

$$\iiint_D f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

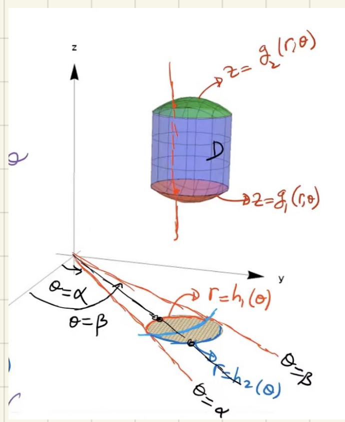
$$\iiint_D f(r,\theta,z)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\,\Delta V_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, r_k\,\Delta r_k\,\Delta\theta_k\,\Delta z_k$$



$$\iiint_D f(x,y,z)\,dV = \iiint_D f(r,\theta,z)\, r\,dr\,d\theta\,dz$$

$$\iiint_D f(x,y,z)\,dV = \iiint_D f(r,\theta,z)\, r\,dz\,dr\,d\theta$$



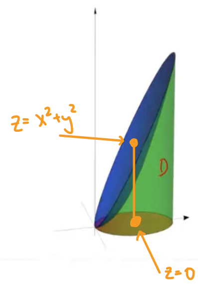
- ☐ z goes from $z = g_1(r,\theta)$ to $z = g_2(r,\theta)$
- ☐ θ goes from $\theta = \alpha$ to $\theta = \beta$
- ☐ r goes from $r = h_1(\theta)$ to $r = h_2(\theta)$

$$\iiint_D f(x,y,z)\,dV$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\, r\,dz\,dr\,d\theta$$

Example-1: Cylindrical Coordinates

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



$$\iiint_D f(r,\theta,z)\, dV$$

$$= \iiint f(r,\theta,z)\, r\, dr\, d\theta\, dz$$

z goes from $z=0$ to $z = x^2+y^2 = r^2$

Here $g_1(r,\theta) = 0$, $g_2(r,\theta) = r^2$

r goes from $r=0$ to $r = 2\sin\theta$

θ goes from 0 to π.

We have $x^2 + (y-1)^2 = 1$

$\Rightarrow x^2 + y^2 - 2y + 1 = 1$

$\Rightarrow x^2 + y^2 - 2y = 0$

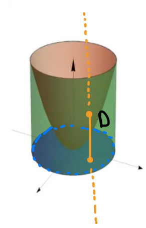
$\Rightarrow r^2 - 2r\sin\theta = 0$

$\Rightarrow r = 2\sin\theta$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=r^2} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

Example-2: Cylindrical Coordinates

Find the centroid $(\delta = 1)$ of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the $xy-$plane.



Using the symmetry, $\bar{x} = \bar{y} = 0$.

$$\text{Mass} = \iiint \delta(x,y,z)\, dV = \iiint (1)\, dV$$

Our z goes from $z=0$ to $z = x^2+y^2 = r^2$

Our r goes from $r=0$ to $r=2$

Our θ goes from $\theta=0$ to $\theta = 2\pi$

$$\text{Mass} = \iiint (1)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \int_0^{r^2} (1)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3\, dr\, d\theta$$

$$= 2\pi \int_0^2 r^3\, dr$$

$$\boxed{= 8\pi}$$

Δ

Δ Now the z-coordinate of the centroid is:

$$\bar{z} = \frac{1}{M} \iiint z\, \delta(x,y,z)\, dV$$

$$= \frac{1}{8\pi} \int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, dz\, r\, dr\, d\theta$$

$$= \frac{1}{8\pi}(2\pi) \int_0^2 \frac{z^2}{2}\Big|_0^{r^2} r\, dr$$

$$= \frac{1}{4} \int_0^2 \frac{r^5}{2}\, dr = \frac{1}{8} \frac{r^6}{6}\Big|_0^2 \boxed{= 4/3}$$

$$\boxed{\text{The centroid is } (0, 0, 4/3)}$$

Note: Throughout this section there is a bit of a subtle notation abuse. If you have a function $f(x,y)$ or $f(x,y,z)$, etc., then when you change coordinates you must make the appropriate substitutions for the variables inside the function, as well. For example, if you have $f(x,y) = xy$, then changing to polar coordinates doesn't mean you get (as could be misconstructed) $f(r,\theta) = r\theta$; you actually get

$$f(x,y) = f(r\cos\theta, r\sin\theta) = r^2\cos(\theta)\sin(\theta).$$

This issue mainly arises when writing down general formulas (because the notation is weird) and not when you're actually evaluating integrals, thankfully, but it's helpful to be aware.

- ☐ Convert from spherical to Cartesian coordinates
- ☐ Compute the volume of a region using a triple integral in Spherical Coordinates

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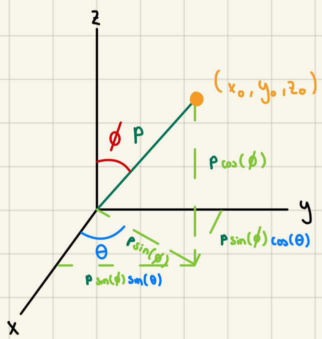
Introduction to Spherical Coordinates



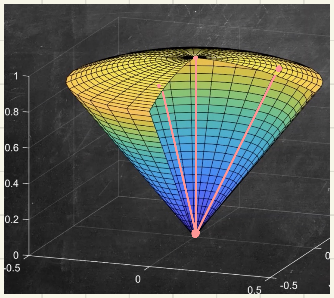
Integral Under Spherical Function $f(\rho,\phi,\theta)$

$$\iiint_R f(\rho,\phi,\theta)\,dV$$

$$\boxed{= \iiint_R f(\rho,\phi,\theta)\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta}$$

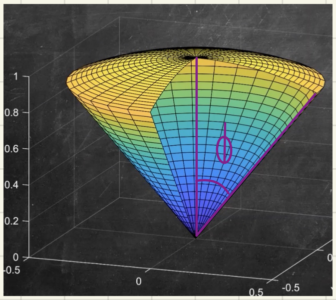
Spherical → Cartesian

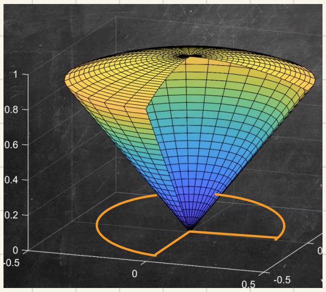
$$x = \rho\sin(\phi)\cos(\theta)$$
$$y = \rho\sin(\phi)\sin(\theta)$$
$$z = \rho\cos(\phi)$$



Region inside of radius $\rho = 1$, with

$0 \le \phi \le \pi/6$ and $0 \le \theta \le 3\pi/2$





$$V = \int_{\theta=0}^{\theta=3\pi/2}\int_{\phi=0}^{\phi=\pi/6}\int_{\rho=0}^{\rho=1} \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

- [ ] Convert equations from spherical coordinates to Cartesian and vice versa
- [ ] Compute triple integrals in cylindrical coordinates

More on Spherical Coordinates

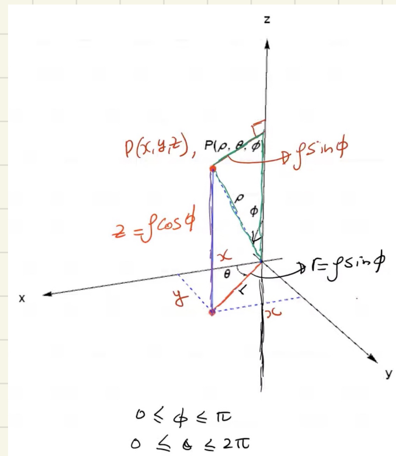
Spherical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

Therefore..

$$\circledast \begin{cases} x = \rho\sin\phi\cos\theta & ;\ \text{where } 0 \le \phi \le \pi \\ y = \rho\sin\phi\sin\theta & \phantom{;\ \text{where }} 0 \le \theta \le 2\pi \\ z = \rho\cos\phi \end{cases}$$

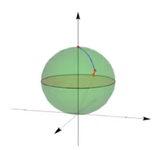
$x^2 + y^2 + z^2 = \rho^2$; Sphere of radius of ρ.



Example-3: Spherical Coordinates

Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z-1)^2 = 1$$



We have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$\Rightarrow \rho^2\sin^2\phi\cos\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Rightarrow \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi \cancel{+1} = \cancel{1}$$

$$\Rightarrow \rho^2 - 2\rho\cos\phi = 0, \quad \rho \neq 0.$$

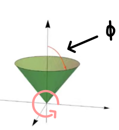
$$\boxed{\Rightarrow \rho = 2\cos\phi}$$

Example-4: Spherical Coordinates

Find a spherical coordinate equation for the cone

$$z = \sqrt{x^2 + y^2}$$

We have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$



$\phi = \pi/4$

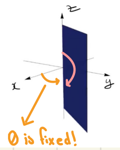
Note: This argument works because ϕ is between 0 and π, so that $\sin(\phi)$ is non-negative.

$$\Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$
$$= \sqrt{\rho^2\sin^2\phi}$$
$$\Rightarrow \rho\cos\phi = \rho\sin\phi \quad \underline{\text{or}} \quad \tan\phi = 1$$

$$\boxed{\Rightarrow \phi = \pi/4}$$

Example: Spherical Coordinates

Find a spherical coordinate equation for the plane shown below.



We draw this plane in spherical coordinates by using $\theta = \pi/3$.

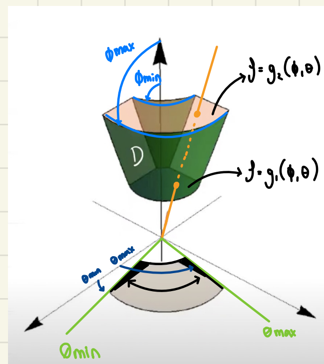
θ = Constant, represents a plane.

ϕ = Constant, represents a cone.

ρ = constant, represents a sphere.

$x^2 + y^2 + z^2 = \rho^2$

$$\iiint_D f(x,y,z)\, dV = \iiint_D f(\rho,\phi,\theta)\, \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$



ρ goes from $g_1(\phi,\theta)$ to $g_2(\phi,\theta)$.

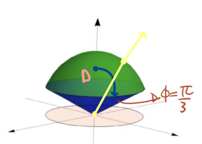
ϕ goes from $\phi = \phi_{min}$ to $\phi = \phi_{max}$.

θ goes from $\theta = \theta_{min}$ to $\theta = \theta_{max}$.

$$= \int_{\theta=\theta_{min}}^{\theta=\theta_{max}} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho,\phi,\theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Example-5: Spherical Coordinates

Find the volume of the "ice-cream cone" D cut from the solid sphere $\rho < 1$ by the cone $\phi = \pi/3$.



$\phi = \frac{\pi}{3}$ represents the cone (blue).

$\rho < 1$ brings us inside the unit sphere.

Volume $= \iiint_D (1)\, dV$.

ϕ goes from 0 to $\pi/3$.

ρ goes from 0 to 1.

θ goes from 0 to 2π.

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 (1)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3} \sin\phi\right]_0^1 d\phi\, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi\, d\phi\, d\theta$$

$$= \frac{2\pi}{3} \int_0^{\pi/3} \sin\phi\, d\phi$$

$$\boxed{= \pi/3}$$

Summary

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Quiz on 15.7

100%

1. Point $(\rho, \phi, \theta) = (4, \pi/3, \pi/6)$.

The x-coordinate of this point represented in Cartesian coordinate is:

$$x = \rho \sin\phi \cos\theta$$
$$= (4)\sin(\pi/3)\cos(\pi/6)$$
$$\boxed{= 3}$$

2. The integral

$$\frac{1}{\pi}\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1+\cos\theta} 4r \, dr \, d\theta \, dz$$

$$\boxed{= 12}$$

3. $4/\pi \int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\sec\phi} (\rho^3 \sin\phi \cos\phi) \, d\rho \, d\phi \, d\theta$

$$\boxed{= 1}$$

15.8 — Change of Variables

- ☐ Correctly write down the Jacobian of a 2D change of variables
- ☐ Compute 2D multivariable integrals using a change of variables

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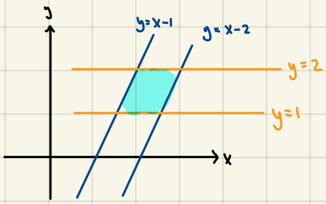
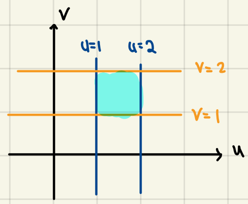
Introduction to Change of Variables



Set $u = x - y$
$v = y$



Substitution in 1D:

For $x = g(u)$, $\displaystyle\int_{x=g(a)}^{x=g(b)} f(x)\,dx = \int_{u=a}^{u=b} f(g(u))\,g'(u)\,du$

Substitution in 2D:

For $x = g(u,v)$, $y = h(u,v)$

$$\iint_R f(x,y)\,dy\,dx = \iint_G f(g(u,v),h(u,v))\,J(u,v)\,du\,dv$$

$$J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$ the "Jacobian"

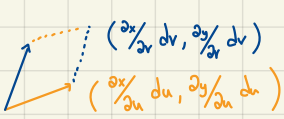
Example: $u = x - y$, $v = y$

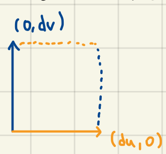
$\Rightarrow x = u + v$, $y = v$

$$J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \iint_R f(x,y)\,dy\,dx = \int_1^2\int_1^2 f(g(u,v),h(u,v))(1)\,du\,dv$$

---

x, y — coordinates $\longleftarrow$ u, v — coordinates :





$$\text{Area} = \left(\tfrac{\partial x}{\partial u}du, \tfrac{\partial y}{\partial u}du\right) \times \left(\tfrac{\partial x}{\partial v}dv, \tfrac{\partial y}{\partial v}dv\right) \qquad \text{Area} = du\,dv$$

$$= \underbrace{\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}}_{J(u,v)} du\,dv$$

- [ ] Sketch regions arising from changes of variables.
- [ ] Compute double integrals via changes of variables

Examples for Double Integrals via Change of Variables

Example-1: Substitution for Double Integrals

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, and write the Cartesian integral $\iint_R f(x,y)\,dx\,dy$ as a polar integral.

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \longrightarrow J(r,\theta) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

Here $u = r$, $v = \theta$,

$$\partial x/\partial r = \cos\theta, \quad \partial x/\partial \theta = -r\sin\theta$$

$$\partial y/\partial r = \sin\theta, \quad \partial y/\partial \theta = r\cos\theta$$

Then

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$x = g(u,v) \text{ and } y = h(u,v)$$

$$\int_R\int f(x,y)\,dx\,dy = \int_G\int f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

for our case!

$$= \int_G\int f(r\cos\theta, r\sin\theta)\,|J(r,\theta)|\,dr\,d\theta$$

no need for the r so it is hidden in the Jacobian!

$$\boxed{= \int_G\int f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Example-2: Substitution for Double Integrals

Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by applying the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}$$

and integrating over an appropriate region in the uv−plane.

From $y/2 = v$

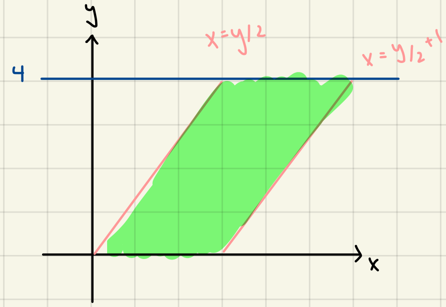
$\Rightarrow y = 2v$

From $u = (2x-y)/2$

$2u = 2x - y$

$\Rightarrow x = u + v$

$\downarrow$

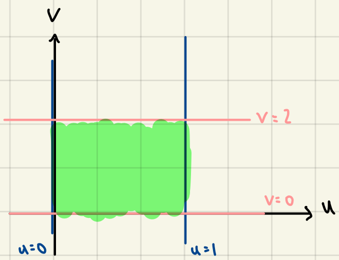


$\longrightarrow$ $\underline{x = y/2}$: $\underline{y=0}$:

$u+v = x = y/2 = v, \ u=0$ $v = 0/2, \ v=0$

$\underline{x = y/2 + 1}$: $\underline{y=4}$:

$u+v = y/2 = v+1, \ u=1$ $v = 4/2, \ v=2$



$$\longrightarrow \quad J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\int_R\int f(x,y)\,dx\,dy = \int_G\int f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv.$$

$$\int_0^4\int_{y/2}^{(y/2)+1} (2x-y)/2 \; dx\,dy = \int_{v=0}^{v=2}\int_{u=0}^{u=1} ((2(u+v)-2v)2)(2)\,du\,dv$$

$$= \int_0^2\int_0^1 (2u)\,du\,dv$$

$$= \int_0^2 \left[u^2\right]_0^1 dv$$

$$= \int_0^2 (1)\,dv \boxed{= 2}$$

Example-3: Substitution for Double Integrals

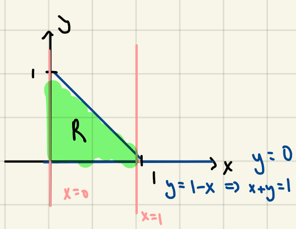
Evaluate

$$\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

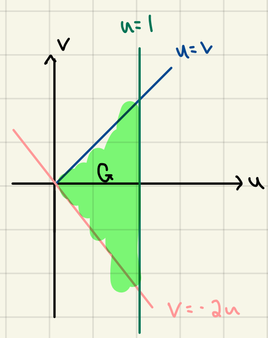
Taking $u = x+y, \ v = y-2x$

$2u = 2x+2y \Rightarrow 2u+v = 3y \rightarrow \boxed{y = 2u/3 + v/3}$

$v = y-2x$ $\boxed{x = u/3 - v/3}$



$\longrightarrow$



$\underline{x=0}$: $u/3 - v/3 = 0 \Rightarrow u=v$

$\underline{y=0}$: $2u/3 + v/3 = 0 \Rightarrow v = -2u$

$\underline{y+x=1}$: $u = x+y = 1 \Rightarrow u=1$

$$J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = 1/9 + 2/9 = 1/3$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy\, dx$$

$$= \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} \sqrt{u}\,(v)^2 (1/3)\, dv\, du$$

$$= 1/3 \int_0^1 \sqrt{u} \left[v^3/3 \right]_{-2u}^{u} du$$

$$= 1/9 \int_0^1 u^{1/2}(u^3 + 8u^3)\, du$$

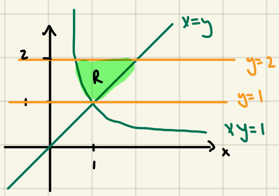
$$= \int_0^1 u^{7/2}\, du \quad \boxed{= 2/9}$$

Example-4: Substitution for Double Integrals

Evaluate

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx dy$$

Here $u = \sqrt{xy}$, $v = \sqrt{y/x}$ $\qquad u^2 = xy$, $v^2 = y/x$ $\qquad$ Similarly:



$xy = 1$ $\qquad \Rightarrow u^2 = x(v^2 x)$ $\qquad y = uv$

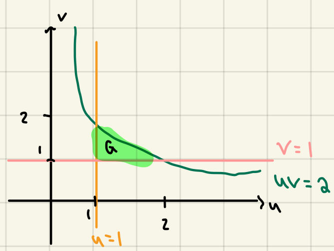
$x^2 = 1$ $\qquad \Rightarrow u^2/v^2 = x^2$,

$x = 1$ $\qquad x = u/v$

$\underline{x=y}$: $u^2 = xy = u^2/v^2 \Rightarrow v = 1$

$\underline{xy=1}$: $u = \sqrt{xy} \Rightarrow u = 1$

$\underline{y=2}$: $uv = 2$, $v = 2/u$



$$J(u,v) = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = u/v + u/v = 2(u/v)$$

$$\int_1^2 \int_{1/y}^{y} \sqrt{x/y}\, e^{\sqrt{xy}}\, dx\, dy$$

$$= \int_{v=1}^{v=2} \int_{u=1}^{u=2/v} v e^u \cdot 2u/v \, du\, dv$$

$$= \int_{u=1}^{u=2} \int_{v=1}^{v=2/u} (2u) e^u \, dv\, du$$

$$= 2 \int_{u=1}^{u=2} u e^u \left[v \right]_1^{2/u} du$$

$$\rightarrow 2 \int_1^2 2e^u - ue^u)\, du$$

$$= 2 \int_1^2 (2-u) e^u\, du \quad \boxed{= 2e(e-2)}$$

- ☐ Explain where the scale factors in cylindrical and spherical coordinates come from
- ☐ Compute triple integrals via changes of variables

Triple Integrals via Changes of Variables

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(u,v,w)|J(u,v,w)|\,du\,dv\,dw, \quad J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$; $dV = r\,dr\,d\theta\,dz \longrightarrow dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$$J(r,\theta,z) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$

$$J(\rho,\phi,\theta) = \begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{vmatrix} = \rho^2\sin\phi$$

Example-5: Substitution for Triple Integrals

Evaluate

$$\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

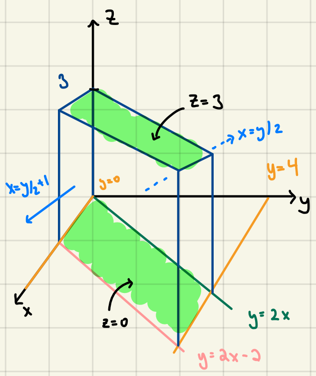
by applying the transformation

$$u=(2x-y)/2, \quad v=y/2, \quad w=z/3$$

and integrating over an appropriate region in the $uvw-$space.

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$z = 3w,\ y = 2v,\ x = u+v$



$\underline{x = y/2}$: $u+v = y/2 = v \Rightarrow u=0$

$\underline{x = y/2+1}$: $u+v = y/2+1 = v+1 \Rightarrow u=1$

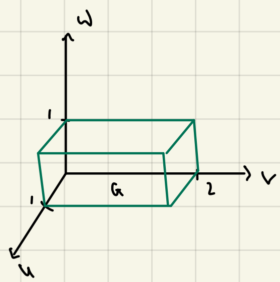
$\underline{y=0}$: $2v = 0 \Rightarrow v=0$

$\underline{y=4}$: $2v=4 \Rightarrow v=2$

$\underline{z=3}$: $3w=3 \Rightarrow w=1$

$\underline{z=0}$: $\Rightarrow w=0.$

$$J(u,v,3) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$



$$\int_0^3\int_0^4\int_{y/2}^{y/2+1}\left(\frac{2x-y}{2}+z/3\right)dx\,dy\,dz$$

$$= \int_{w=0}^{w=1}\int_{v=0}^{v=2}\int_{u=0}^{u=1}(u+w)(|6|)\,du\,dv\,dw$$

$$= 6\int_0^1\int_0^2\int_0^1 (u+w)\,du\,dv\,dw$$

$$\rightarrow 6\int_0^1\int_0^2 (u^2/2 + uw)\Big|_0^1 dv\,dw$$

$$= 6\int_0^1\int_0^2 (1/2+w)\,dv\,dw$$

$$\boxed{=12}$$

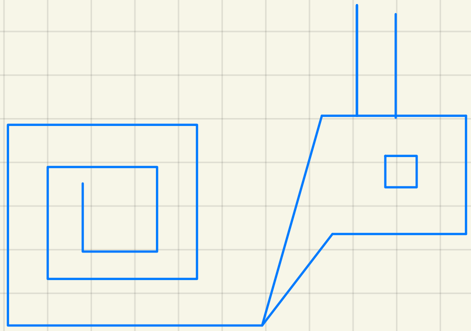
Quiz on 15.8

100%

1. $u = 2x - y$, $v = 2y$; compute $J(u,v)$

$y = v/2$ $\quad x = \frac{u+y}{2} = \frac{u + v/2}{2}$

$$\begin{vmatrix} 1/2 & 1/4 \\ 0 & 1/2 \end{vmatrix} = (1/2)(1/2) - (0)(1/4) = 1/4 \ \boxed{= 0.25}$$



2. $u = x + y + z$, $v = x + y - z$, $w = x - y + z$

$x = u - y - z$ $\quad y = v - x + z$ $\quad z = w - x + y$

$x + y = u - z$

$z = w - (u - z)$

$z = w - u + z$

$u = u$

$V = (u - y - z) + y - (w - x + y)$

$V = u - y - z + y - w + x - y$

$V = u - w - y - z + x$

$$\begin{vmatrix} x+y+z & u \\ x+y-z & v \\ x-y+z & w \end{vmatrix} = 0$$

$u + v = x + y + z + x + y - z$

$u + v = 2x + 2y$

$\frac{u+v}{2} = x + y$

$u - v = 2z$

$\frac{u-v}{2} = z$

$u = \frac{u+v}{2} + z$

$u - \left(\frac{u+v}{2}\right) = z$

$V = x + y - z - (u - y - z)$

$V = x + y - z - u + y + z$

$V = x + 2y - u$

$w - \frac{u+v}{2} = z$

$\boxed{= -0.25}$

Guessed the answer!

$u + v + w = x + y + z + x + y - z + x - y + z$

$u + v + w = 3x + 1y + 1z$