# 14.1 – Contour Plots

- Convert between a contour plot and the graph of a function

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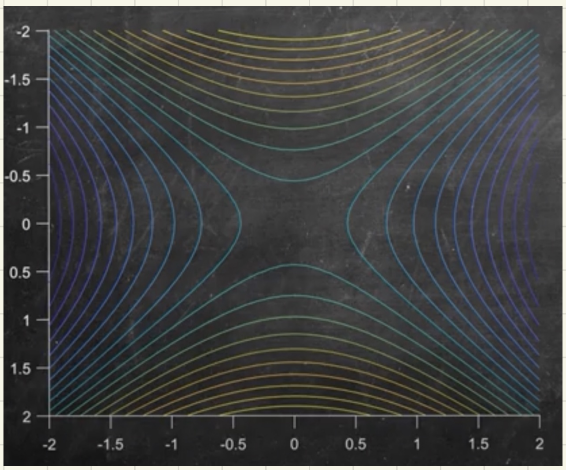
## Multi-Variable Functions & Contour Plots

Graph of $z = f(x,y) = x^2 + y^2$

One output — Two Inputs

A contour is a fixed height $f(x,y) = c$

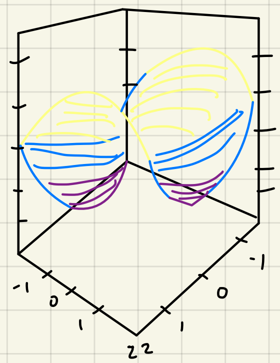
Example:




Blue: lower values

Yellow: higher values

$z = f(x,y) = x^2 - y^2$




- We define Interior, Boundary and Exterior points, both for 2D and 3D regions.
- Examples on the domain and range of functions that depend on two or three variables.

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## Partial Derivatives: Functions of Several Variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A Real-Valued Function $f$ on D is a rule that assigns a unique value

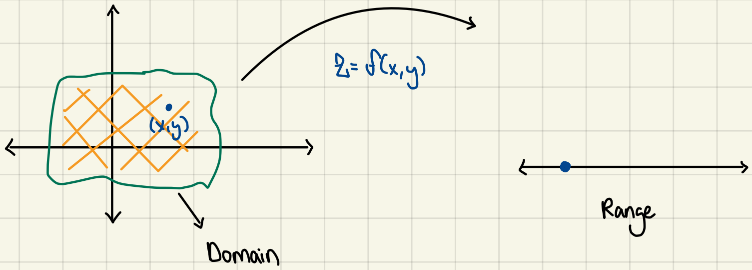
Dependent Variable → $w = f(\underbrace{x_1, x_2, \dots, x_n}_{\text{Independent Variables}})$ $\qquad y = f(x)$

to each element in D.

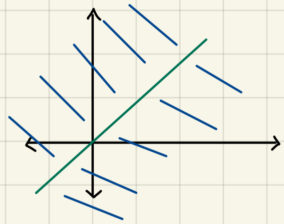
The set D is the Domain of the function $f$. The set of w-values that $f$ gives is the function's Range.

↓

Consider first $\quad w = f(x_1, x_2)$ , $\underbrace{z = f(x,y)}$




Example: ① $z = 1/x-y$ ; Domain: $y \neq x$.



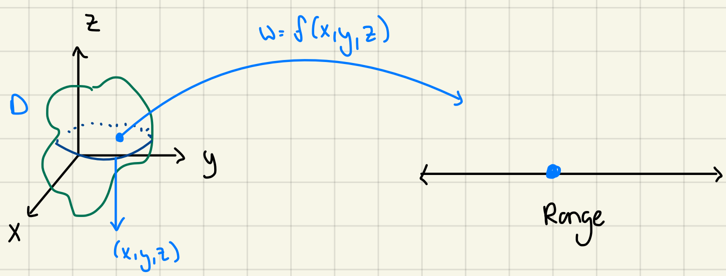
② $z = +\sqrt{y - x^2}$ ; Domain: $y - x^2 \geq 0$

Range: $[0, \infty)$ $\qquad$ or $y \geq x^2$

③ $z = \sin(xy) = f(x,y)$ ; Domain: $\mathbb{R}^2$

Range: $[-1, +1]$

Now Consider $\quad W = f(x_1, x_2, x_3)$ ; $w = f(x,y,z)$ $\qquad (x,y,z) \in D.$




Example: ① $w = \sqrt{x^2 + y^2 + z^2}$ ; domain $= \mathbb{R}^3$

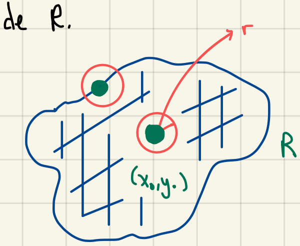
range $= [0, \infty)$

② $w = xy\ln(z)$ ; domain: $z > 0$, Everything above the xy-plane

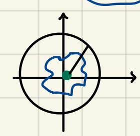
range: $(-\infty, \infty)$

**Definition**: A point $(x_0, y_0)$ in a region $R$ in $\mathbb{R}^2$ is an Interior Point of $R$ if it is the centre of a disk of positive radius, that lies completely inside $R$.

$(x_0, y_0)$ is a Boundary Point of $R$ if every disk that we can centre at $(x_0, y_0)$ contains points that are inside $R$ aswell as the point that lies outside $R$.



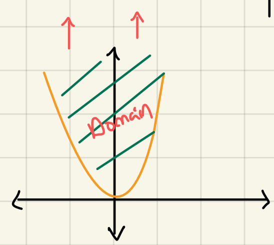

Further, A region in the plane is Bounded if it lies inside a disk of finite radius.



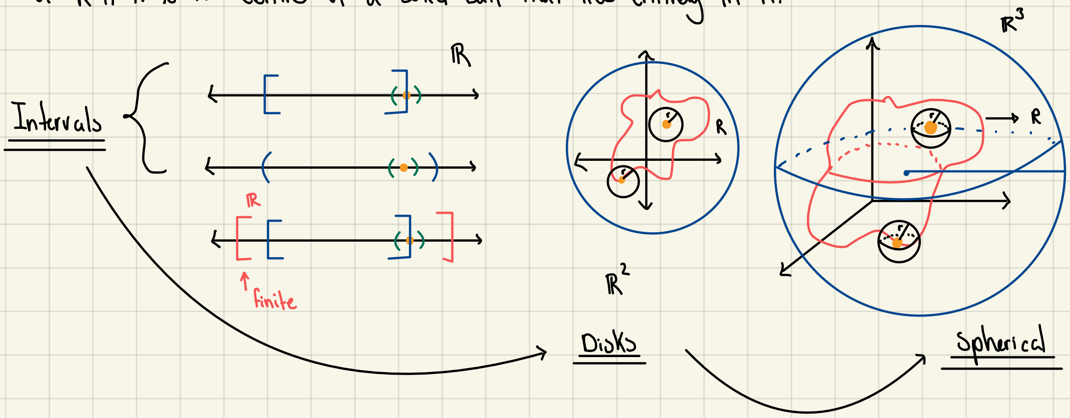
Example: $z = f(x,y) = \sqrt{y - x^2}$

Domain: $y \geq x^2$, Range $= [0, \infty)$

It is unbounded!




**Definition**: A point $(x_0, y_0, z_0)$ in a region $R$ in space $(\mathbb{R}^3)$ is an Interior Point of $R$ if it is the centre of a solid ball that lies entirely in $R$.




- We define and look at an example of Level curves and Contour Curves
- Further, we work out an example on Level surfaces

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## Functions of Several Variables

Example: $z = f(x,y) = 100 - x^2 - y^2$, $z \geq 0$

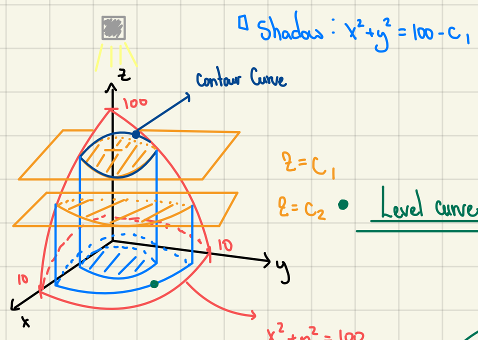
For $z = 0$, $x^2 + y^2 = 100$

$z = 100 - x^2 - y^2$ take $z = c$

$\Rightarrow 100 - x^2 - y^2 = c_1$

$(100 - c) = x^2 + y^2$

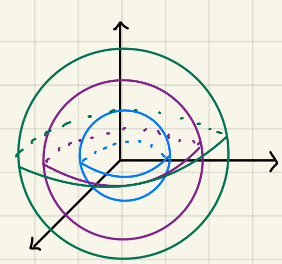
radius $= \sqrt{100 - c_1}$




Suppose $w = f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$

Setting $w = c$

$\Rightarrow x^2 + y^2 + z^2 = c^2$ Level Surfaces



# Quiz on 14.1

$\boxed{100\%}$

1. $z = f(x,y) = x^2 - y^2$ @ $z=0$

What do th level curves look like?

× ☐ Hyperbola ✓☐ Two Intersecting Lines

2. Domain & Range: $z = f(x,y) = \sqrt{x} + \sqrt{y}$

✓☐ First Quadrant, $[0, \infty)$ $x, y \geq 0$

3. Level Surface $h(x,y,z) = x^2 + y^2$

$h(x,y,z) = 4$ $x^2 + y^2 = 4$ $x^2 + y^2 - 4 = 0$

✓☐ An infinite vertical Cylinder of radius 2

# 14.2 — Limits of Multivariable Functions

- [ ] Demonstrate a limit does not exist by finding two appropriate paths.

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## Limits of Multivariable Functions

$$f(x,y) = \frac{xy}{x^2+y^2} \leftarrow \text{Problematic at } (0,0)!$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ???$$

- Restrict to line $y=x$

$$= \frac{xy}{x^2+y^2} \Rightarrow f(x) = \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} \longrightarrow \lim_{x\to 0} \frac{x^2}{2x^2} = 1/2$$

- Restrict to line $y=-x$

$$= \frac{xy}{x^2+y^2} \Rightarrow f(x) = \frac{-x^2}{x^2+x^2} = \frac{-x^2}{2x^2} \longrightarrow \lim_{x\to 0} \frac{-x^2}{2x^2} = -1/2$$

> If $f(x,y)$ has two different limits along two different paths $(x,y) \to (x_0,y_0)$ then
> $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ does not exist.}$$

Question 1: How do we define a limit existing at a point?

Question 2: Suppose that approaching along any STRAIGHT line gives the same value. Does the limit necessarily exist?

- [ ] State the common Limit Laws for multivariable functions
- [ ] Apply algebraic tricks like factoring and radical conjugate to solve a multivariable limit.

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## Algebraically Showing Limits Exist

Example $\lim_{(x,y)\to(1,2)} \frac{3x^2+1}{2y} = \frac{3\cdot 1+1}{2\cdot 2} = 1$

$\downarrow$

**Suppose**: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$, $K \in \mathbb{R}$

- $\lim_{(x,y)\to(a,b)} \left(f(x,y) + g(x,y)\right) = L + M$
- $\lim_{(x,y)\to(a,b)} \left(f(x,y) \cdot g(x,y)\right) = L \cdot M$
- $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ if $M \neq 0$
- $\lim_{(x,y)\to(a,b)} K f(x,y) = KL$

**Suppose**: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $g(x)$ continuous at $L$

Then: $\lim_{(x,y)\to(a,b)} g(f(x,y)) = g(L)$

**Example** $\lim_{(x,y)\to(3,2)} \dfrac{xy^2 - 3y^2 + 2x - 6}{3y - xy}$

$$= \lim_{(x,y)\to(3,2)} \frac{xy^2 - 3y^2 + 2x - 6}{y(3-x)}$$

$$= \lim_{(x,y)\to(3,2)} \frac{(y^2+2)\cancel{(x-3)}}{y\cancel{(3-x)}}$$

$$= \lim_{(x,y)\to(3,2)} \frac{-(y^2+2)}{y} = -3$$

**Example** $\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x+y-4}{\sqrt{x+y}-2} \cdot \left(\dfrac{\sqrt{x+y}+2}{\sqrt{x+y}+2}\right)$

$$= \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{\cancel{(x+y-4)}(\sqrt{x+y}+2)}{\cancel{(x+y-4)}}$$

$$= \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \sqrt{x+y}+2$$

$$= \sqrt{2+2} + 2 = 4$$

**Example** $\lim_{(x,y)\to(0,\pi)} \sin(x^2+y)$

$$= \sin(\pi) = 0$$

- ☐ Epsilon-Delta definition of limit
- ☐ Basic laws of limit

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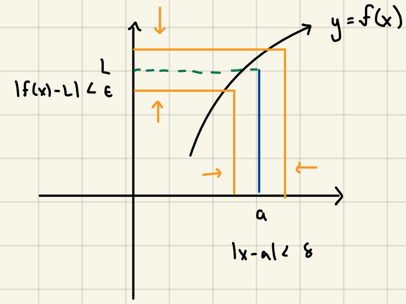
## Limits & Continuity in Higher Dimensions | Part 1

$$\lim_{x\to a} f(x) = L.$$

The limit of $f(x)$ as $x$ approaches $a$ is the number $L$, i.e.,




△ $\lim x\to a\ f(x) = L$.

If, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

Now lets take a function that depends on two variables; $f(x,y)$.

We say that a function $f(x,y)$ approaches the limit $L$ as $(x,y)$ approaches $(x_0,y_0)$, i.e.,

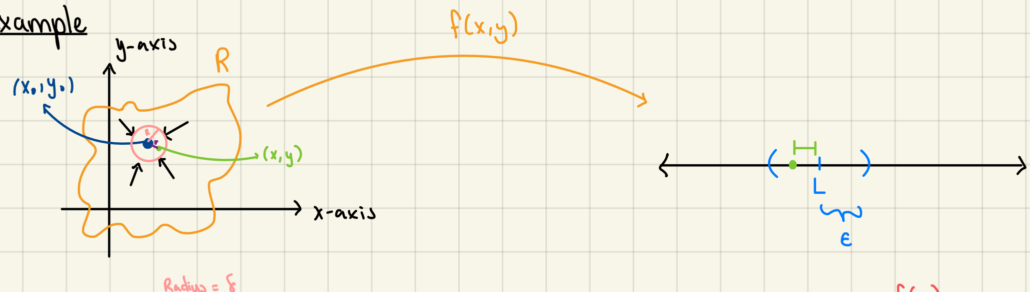
$$\lim_{\underbrace{(x,y)\to(x_0,y_0)}_{\text{NEW!}}} f(x,y) = L$$

if for every $\epsilon > 0$, there exists a corresponding $\delta > 0$, such that for all $(x,y)$ in the domain of $f$...

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{} < \delta$$

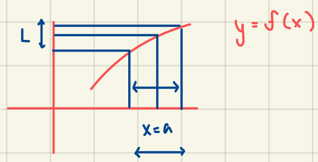
△ $|f(x) - L| < \epsilon$ whenever previously on MATH 100... $0 < |x-a| < \delta$

### Example




$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Note :



# Limit Laws

Suppose $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$ , $K \in \mathbb{R}$.

1. $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L \pm M.$

2. $\lim_{(x,y)\to(x_0,y_0)} K f(x,y) = KL$

3. $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = LM.$

4. $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ ; $M \neq 0$.

5. $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$ ; $n$ is a positive integer.

6. $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$ or $L^{1/n}$ ; $n$ is a positive integer and if $n$ is even then $L$ must be positive.

Example ① $\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$ , $x \to 0$, $y \to 1$

$= \frac{\lim_* [x - xy + 3]}{\lim_* [x^2y + 5xy - y^3]}$

$= \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (3)^3}$

$= 3/-1^3 = -3$

② $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

$= \sqrt{\lim_* x^2+y^2}$

$= \sqrt{(3)^2 + (-4)^2}$

$= \sqrt{25} = 5$

③ $\lim_{(x,y)\to(1,1)} \left(\frac{y}{x}\right)$

$= \frac{\lim_* (y)}{\lim_* (x)}$

$= 1/1 = 1$

- ☐ Examples on algebraic methods for computation of limits
- ☐ Examples on Two-Path method for disproving a limit

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## Limits & Continuity in Higher Dimensions | Part 2

**Example** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

$$= \lim \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}\left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$$

$$= \lim \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x})^2-(\sqrt{y})^2}$$

$$= \lim \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$$

$$= 0$$

**Example** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Using Polar Coordinates ... $x = r\cos\theta$, $y = r\sin\theta$

$\boxed{r^2 = x^2+y^2}$ $(x,y)\to 0$ give $r\to 0$

$$= \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \lim \frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}}$$

$$= \lim (r\cos\theta\sin\theta) = 0.$$

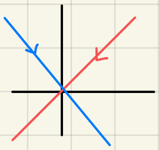
**Example** $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x}{y}\right)$ DNE

$x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \lim_{r\to 0}\left(\frac{r\cos\theta}{r\sin\theta}\right) = \lim_{r\to 0}(\tan\theta). \quad \text{DNE.}$$

Suppose we approach $(0,0)$ through the line $y=x$

$$\lim (y/x) = \lim (x/x) = 1.$$

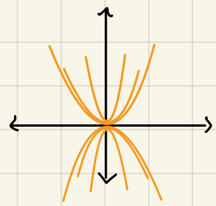


Now lets approach through $y=-x$.

$$\lim (y/x) = \lim (-x/x) = -1.$$

$\Rightarrow$, DNE

Example $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ DNE



Lets approach (0,0) through Parabolas $y = kx^2$

$$\lim_{\to} \frac{2x^2y}{x^4+y^2}$$

$$= \lim_{\to} \frac{2x^2(Kx^2)}{x^4+(Kx^2)^2}$$

$$= \lim_{x\to 0} \frac{2Kx^4}{x^4+K^2x^4}$$

$$= \frac{2K}{1+K^2}$$

For $K=1$, $\lim_{\to} \frac{2x^2y}{x^4+y^2} = 1$

$K=2$, $\lim_{\to} \frac{2x^2y}{x^4+y^2} = 4/5$

Note: * Having the same limit along, for example, all straight lines approaching $(x_0, y_0)$ DOES NOT imply that a limit exists at $(x_0, y_0)$.

Example $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

using Polar!

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{\to} (\cos^2\theta - \sin^2\theta)$$

$$= \cos(2\theta)$$, DNE

| |
|---|
| Two-Path Method is only used to prove the non-existence of a limit at $(x_0, y_0)$. |

□ Continuity of a function of two-variables

Limits & Continuity in Higher Dimensions | Part 3

# Continuity:

A function $f$ is continous at $x = a$ if...

① $\lim_{x \to a} f(x)$ exists.

② $f$ is well defined at $a$.

③ $\boxed{\lim_{x \to a} f(x) = f(a).}$

A function $f$ is continuous at $(x_0, y_0)$ if...

① $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ exists.

② $f(x,y)$ is well defined at $(x_0, y_0)$.

③ $\boxed{\lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0, y_0)}$

## Example

$$f(x,y) = \begin{cases} 2xy / x^2+y^2 & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Discuss the continuity at $(0,0)$.

① Is $f(x,y)$ defined at $(0,0)$? Yes!

② $\lim_{(x,y) \to (0,0)} f(x,y)$? DNE

$\lim_{*} f(x,y) = \lim_{*} 2xy / x^2+y^2$

using Polars:

$$= \lim_{r \to 0} \frac{2r^2 \cos\theta \sin\theta}{r^2(\cos^2\theta + \sin^2\theta)}$$

$= \lim_{r \to 0} \sin(2\theta)$

$= \sin(2\theta)$ DNE

$f(x,y)$ is continuous at every point except at $(0,0)$.

## Continuity of Composite Functions

If $f(x,y)$ is continuous at $(x_0, y_0)$ and $g(x)$ is a single variable function that is continuous at $f(x_0, y_0)$, then the composition of functions

at $f(x_0, y_0)$

$h = g \circ f = g(\underbrace{f(x,y)})$

is continuous at $(x_0, y_0)$

**Example:** $f(x,y) = x - y$, $g(x) = e^x$.
$h = g \circ f = g(f(x,y)) = e^{x-y}$, continous.

$f(x,y) = xy / x^2+1$, $g(x) = \sin x$
$h = g \circ f = g(f(x,y)) = \sin(xy / x^2+1)$; continuous

# Quiz on 14.2

✓ 1. $\lim_{(x,y)\to(1,2)} \dfrac{-yx^2 + x^2 - 2y + 2}{yx - x} = -3/1 = -3$

✓ 2. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{2x^2+3y^2} = \dfrac{0}{0}$

$$\frac{(r\cos\theta)(r\sin\theta)}{2r^2\cos^2\theta + 3r^2\sin\theta} = \frac{r(\cos\theta)(\sin\theta)}{(2r^2+3r^2)(\cos^2\theta+\sin^2\theta)} = \frac{\cos\theta\sin\theta}{5r} = \text{DNE}$$

$y = x \Rightarrow \dfrac{\cancel{x^2}}{5\cancel{x^2}} = 1/5$

✓ 3. $f(x,y,z) = \dfrac{x^2+1}{y^2+z^2-1}$

All points that are not on th infinite Cylinder $y^2+z^2=1$

# 14.3 – Partial Derivatives

- [ ] Visualize Partial Derivatives
- [ ] Compute Partial Derivatives

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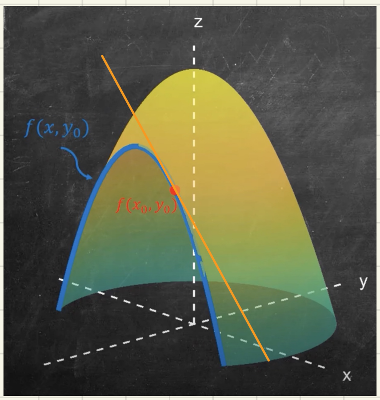
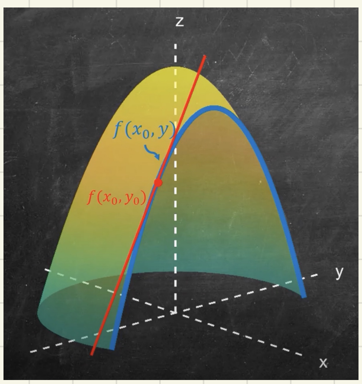
## Partial Derivatives




$f(x, y_0)$ is a one-dimensional function.

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx}\overbrace{(f(x,y_0))}^{\text{Just depends on } x.}\right|_{x=x_0}$$

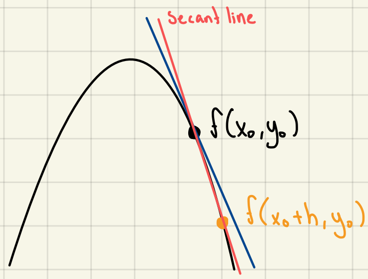
Partial of f w/ respect to x.
or del f over del x.




$f(x_0, y)$ is a one-dimensional function.

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy}\overbrace{(f(x_0,y))}^{\text{Just depends on } y}\right|_{y=y_0}$$

ex.




$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx}(f(x,y_0))\right|_{x=x_0}$$

$$= \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

ex. $\frac{\partial}{\partial x}[3x^2y^3] = 6xy^3$

y is a constant w/ respect to x.

What about other directions than parallel to x and y?

- ☐ Computing Partial Derivatives using Limit definition and Rules of Differentiation
- ☐ Partial Derivatives and Implicit Differentiation
- ☐ Mixed Partials (Higher Order Derivatives)

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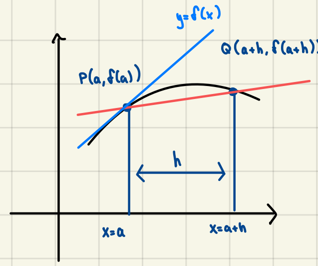
## Partial Derivatives, Implicit Differentiation, and Higher Order Derivatives

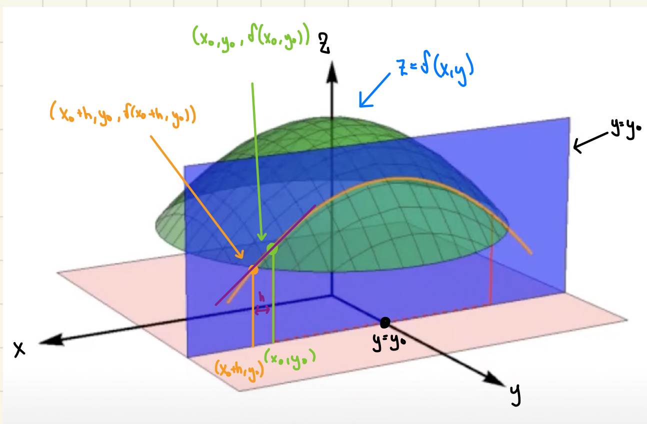
For a function $y=f(x)$, we had...




$$dy/dx = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

And...

$$dy/dx \Big|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$




$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

Note: $$\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

Notations: $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$ , $\frac{\partial}{\partial x}(x_0,y_0)$ , $f_x\Big|_{(x_0,y_0)}$ , $f_x(x_0,y_0)$

Example $f(x,y) = xy$ , $\partial f/\partial x = ?$ at $(x_0,y_0)$

$$\partial f/\partial x \Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

$$= \lim \frac{(\cancel{x_0}+h)\cdot y_0 - \cancel{x_0 y_0}}{h}$$

$$= \lim \cancel{h}(y_0)/\cancel{h}$$

$$= y_0$$

Similarly $$\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = x_0$$

↓

**Example 1** $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4,-5)$ if $f(x,y) = x^2 + 3xy + y - 1$

$$\partial f/\partial x = \partial/\partial x\,(x^2 + 3xy + y - 1)$$
$$= \partial/\partial x\,(x^2) + \partial/\partial x\,(3xy) + \partial/\partial x\,(y) - \partial/\partial x\,(1)$$
$$= 2x + 3y + 0 + 0$$
$$= 2x + 3y, \quad \Rightarrow \partial f/\partial x \Big|_{(4,-5)} = 2(4) + 3(-5) = 8 - 15 = -7$$

$$\partial f/\partial y \Big|_{(4,-5)} = 13$$

**Example 2** $\partial f/\partial y$ as a function if $f(x,y) = y \sin xy$

$$\partial f/\partial y = \partial/\partial y\,[y \cdot \sin(xy)]$$
$$= \partial/\partial y\,(y) \cdot \sin(xy) + y\,\partial/\partial y\,(\sin(xy))$$ ; Product Rule!
$$= \sin(xy) + y\cos(xy)\,\partial/\partial y\,(xy)$$
$$= \sin(xy) + y\cos(xy) \cdot x$$
$$= \sin(xy) + xy\cos(xy)$$

Note: $d/dy\,(2y)$
$$= 2\,d/dy\,(y)$$
$$= 2$$

**Example 3** find $f_x$ and $f_y$ as a function if $f(x,y) = 2y/(y + \cos x)$

$$\partial f/\partial y = \frac{(y+\cos x)\,d/dy\,(2y) - (2y)\,d/dy\,(\overset{1}{y} + \overset{0}{\cos x})}{(y+\cos x)^2}$$

$$= \frac{(\cancel{y}+\cos x)(2) - (\cancel{2y})(1)}{(y+\cos x)^2}$$

$$\Rightarrow \partial f/\partial y = \frac{2\cos x}{(y+\cos x)^2} \qquad \Rightarrow \partial f/\partial x = \frac{2y\sin x}{(y+\cos x)^2}$$

$$\partial f/\partial x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

**Example 4** $\partial z/\partial x$ if the equation $yz - \ln z = x + y$ defines $z$ as a function of the two independent variables $x$ and $y$ and the partial derivative exists.

$z = f(x,y)$ Explicit!

Taking the derivative of the equation w/ respect to $x$, on both sides.

↓

Remember: $x^2 + y^2 = 1$ ; $y = f(x)$
$$\Rightarrow d/dx\,(x^2 + y^2) = d/dx\,(1)$$
$$\Rightarrow 2x + 2y\,dy/dx = 0$$
$$\Rightarrow dy/dx = -x/y, \quad \tfrac{d}{dx}(y) = dy/dx$$

$\partial/\partial x\,(yz) - \partial/\partial x\,(\ln z) = \partial/\partial x\,(x) + \partial/\partial x\,(y)$

$\Rightarrow y\, dz/dx - \frac{1}{z}\, dz/dx = 1 + 0$

$\Rightarrow dz/dx\,[y - 1/z] = 1$

$\Rightarrow dz/dx = 1/(y - 1/z) = z/(yz - 1)$

**Example 7** $R_1, R_2, R_3$ connected in parallel to make an $R$ th value of $R$...

$R = f(R_1, R_2, R_3)$

$$1/R = 1/R_1 + 1/R_2 + 1/R_3$$

Find $\partial R/\partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$

$\partial/\partial R_2\,(1/R) = \partial/\partial R_2\,\cancel{(1/R_1)}^{\,0} + \partial/\partial R_2\,(1/R_2) + \partial/\partial R_2\,\cancel{(1/R_3)}^{\,0}$

$\Rightarrow -1/R^2 \cdot \partial R/\partial R_2 = -1/R_2^2$

$\Rightarrow \partial R/\partial R_2 = (R/R_2)^2$

$\Rightarrow \partial R/\partial R_2 \,\Big|_{(30,45,90)} = \left[\frac{R|_{(30,45,90)}}{45}\right]^2$

$$\boxed{= 1/9}$$

**Example 9** If $f(x,y) = x\cos y + ye^x$, find the second-order derivatives

$$\partial^2 f/\partial x^2,\quad \partial^2 f/\partial y^2,\quad \partial^2 f/\partial x\partial y,\quad \partial^2 f/\partial y\partial x$$

$\partial^2 f/\partial x^2 = \partial/\partial x\,(\partial f/\partial x)$, $\partial^2 f/\partial y\partial x = \partial/\partial y\,(\partial f/\partial x)$

Note: $f_{yx} = (f_y)_x$ ; $\partial^2 f/\partial y\partial x = \partial/\partial y\,(\partial f/\partial x)$

inner most outer most last one first one

**Theorem** If $f(x,y)$ and its partial derivatives $f_x, f_y, f_{xy}$ and $f_{yx}$ are defined throughout an open region containing $(x_0, y_0)$ and all continuous at $(x_0, y_0)$ then...

$$f_{xy}\Big|_{(x_0,y_0)} = f_{yx}\Big|_{(x_0,y_0)}$$

$\partial f/\partial x = \partial/\partial x\,[x\cos y + ye^x] = \cos y + ye^x$

$\partial f^2/\partial x = \partial/\partial x\,(\cos y) + \partial/\partial x\,(ye^x) = 0 + ye^x = ye^x$

$\longrightarrow f_{xx} = \partial^2 f/\partial x^2 = ye^x$

$\partial^2 f/\partial y\partial x = \partial/\partial y\,(\partial f/\partial x) = \partial/\partial y\,(\cos y + ye^x)$

$= -\sin y + e^x(1)$

$= -\sin y + e^x$

**Example 11** find $f_{yxyz}$ if $f(x,y,z) = 1 - 2xy^2z + x^2y$

$f_y = -4xyz + x^2$

$f_{yx} = \partial/\partial x\,(f_y) = -4yz + 2x$

$f_{yxy} = \partial/\partial y\,(f_{yx}) = -4z + 0 = -4z$

$f_{yxyz} = \partial/\partial z\,(f_{yxy}) = -4$

- ☐ Compute partial derivatives and limits for piecewise-defined functions
- ☐ Contrast the relationship between continuity and partial derivatives w/ that of single variable functions

---

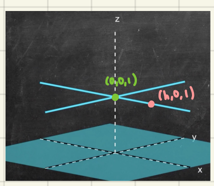
## Continuity vs Partial Derivatives vs Differentiability

### Single Variable:

- ☐ Discontinuous → Not Differentiable

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

### Multivariable:




$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $y=0$

$= \lim 1 = 1$

$\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $y=x$

$= \lim 0 = 0$

$f(0+\Delta x, 0+\Delta y) - f(0,0)$
$= \partial f/\partial x\big|_{(0,0)} \cdot \Delta x + \partial f/\partial y\big|_{(0,0)} \cdot \Delta y + \varepsilon_1(\Delta x) + \varepsilon_2(\Delta y)$

$= 0 + 0 + \varepsilon_1(\Delta x) + \varepsilon_2(\Delta y)$

$0 - 1 = 0 + 0 + \varepsilon_1(\Delta x) + \varepsilon_2(\Delta y)$

$\boxed{\text{So not differentiable!}}$

So discontinuous at $(0,0)$ as the limit has different values along different paths!

---

$$\partial f/\partial x\big|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim (1-1)/h = 0$$

$$\partial f/\partial y\big|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim (1-1)/h = 0$$

$\boxed{\text{So discontinuous at } (0,0)\text{, but partials exists!}}$

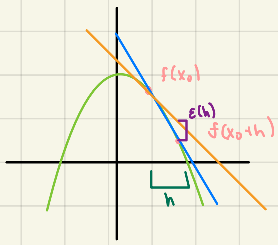
- "Define" Differentiability of a multivariable function [see th note below]
- Illustrate th relationship between differentiability, partial derivatives, and continuity

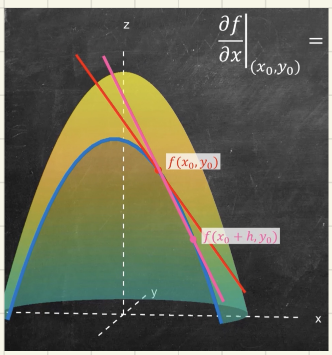
---

## Differentiability of Multivariable functions

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \longrightarrow f(x_0+h) - f(x_0) = \underbrace{\boxed{f'(x_0)h}}_{\text{Approximation}} + \underbrace{\boxed{\varepsilon(h)}}_{\text{Error}}$$

Where $\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$




$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} \longrightarrow f(x_0+\Delta x, y_0) - f(x_0, y_0)$$

$$\underbrace{\boxed{= \partial f/\partial x \,\Big|_{(x_0,y_0)} \cdot \Delta x}}_{\text{Approximation}} + \underbrace{\boxed{\varepsilon_1(\Delta x)}}_{\text{Error}}$$

Note: $f(x_0, y_0+\Delta y) - f(x_0, y_0)$

$$= \partial f/\partial y \,\Big|_{(x_0,y_0)} \cdot \Delta y + \varepsilon_2(\Delta y)$$

$f$ is differentiable at $(x_0, y_0)$ if..

$$\boxed{\begin{aligned} & f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\ & \quad = \partial f/\partial x \,\Big|_{(x_0,y_0)} \cdot \Delta x + \partial f/\partial y \,\Big|_{(x_0,y_0)} \cdot \Delta y + \varepsilon_1(\Delta x) + \varepsilon_2(\Delta y) \end{aligned}}$$

Where $\lim_{\Delta x \to 0} \frac{\varepsilon_1(\Delta x)}{\Delta x} = 0$, $\lim_{\Delta y \to 0} \frac{\varepsilon_2(\Delta y)}{\Delta y} = 0$

**Theorem** If $\partial f/\partial x$ and $\partial f/\partial y$ are continuous on an open region $R$, then $f$ is differentiable on $R$.

# Quiz on 14.3

100%

1. $\partial/\partial y\,[4x^3y^2 + x]$ $(1,3)$

$= 8x^3y$

$= 24$ ✓

✓X 2. $\partial f/\partial y\,\big|_{(x_0,y_0)}$ TRUE?

- [x] Th slope when constrained to a fixed x-value. X ✓
- [ ] Th slope when constrained to a fixed y-value.
- [x] Th slope wh moving parallel to th x-axis.
- [ ] Th slope wh moving parallel to th y-axis. X ✓

3. z function of x and y $\quad x^2 - y^2 + z^2 - 2z = 4$

find $dz/dy\,(2,0,0)$

$\partial/\partial y\,(x^2) - \partial/\partial y\,(y^2) + \partial/\partial y\,(z^2) - \partial/\partial y\,(2z) = \partial/\partial y\,(4)$

$0 - 2y + 2z\, dz/dy - 2\, dz/dy = 0$

$2z\, dz/dy - 2\, dz/dy = 2y$ $\qquad = 0$ ✓

$dz/dy\,[2z - 2] = 2y$

$dz/dy = 2y/(2z-2)$

4. $f(x,y) = (x+y)e^y$ third derivative $\partial^3 f/\partial^2 y\, dx$ at th origin.

$\partial_y(\partial f/\partial x)$

$\partial_y(e^y)$

$\partial_y(e^y)$

$= e^y\,\big|_{(0,0,0)}$

$= 1$ ✓

# 14.4 — Chain Rule

☐ Compute the derivative of a multivariable composition.

---

## Intro to Multivariable Chain Rule

**Remember** 1D Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

**Consider**

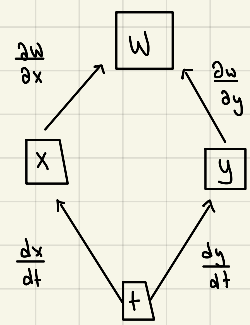
$$W = f(x(t), y(t))$$

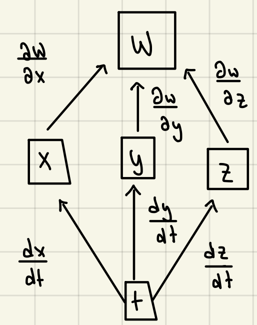
single variable functions

**Chain Rule:**

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

**Dependency Diagram**




Suppose I make a small change $\Delta t$

Get a small change $\Delta x \approx \frac{dx}{dt}\Delta t$

AND a small change $\Delta y \approx \frac{dy}{dt}\Delta t$

$\Delta W \approx \frac{\partial w}{\partial x}\Delta x$ comes from x changes $\longrightarrow$ $\Delta W \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t$ comes from x changes

$\Delta W \approx \frac{\partial w}{\partial y}\Delta y$ comes from y changes $\longrightarrow$ $\Delta W \approx \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t$ comes from y changes

**Example.** $W = f(x,y) = x^2 y$, $x(t) = (2t+1)$, $y(t) = t^3$

$$= 2xy \cdot 2 + x^2 \cdot 3t^2$$
$$= 2(2t+1)t^3 \cdot 2 + (2t+1)^2 \cdot 3t^2$$

☐ Working through examples on Multivariable Chain Rule.

---

# Chain Rule

<u>Example</u>: $w = xy$ , $x = \cos t$ , $y = \sin t$ @ $t = \pi/2$

We have $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$

① $\Rightarrow \frac{\partial w}{\partial x} = y = \sin t$ , $\frac{\partial w}{\partial y} = x = \cos t$ , $\frac{dx}{dt} = -\sin t$ , $dy/dt = \cos t$

Now we get $\frac{dw}{dt} = (\sin t)(-\sin t) + (\cos t)(\cos t)$

$= -\sin^2 t + \cos^2 t$

$\Rightarrow \frac{dw}{dt} = \cos(2t)$

$\Rightarrow \left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos(2 \cdot \pi/2) = -1$

<u>Example</u>: $w = xy + z$ , $x = \cos t$ , $y = \sin t$ , $z = t$ @ $t = 0$

We have $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{dw}{dz} \frac{dz}{dt}$

$\partial w/\partial x = y = \sin t$ , $\partial w/\partial y = x = \cos t$ , $\partial w/\partial z = 1$

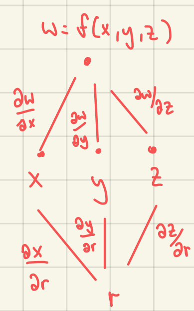
$dx/dt = -\sin t$ , $dy/dt = \cos t$ , $dz/dt = 1$

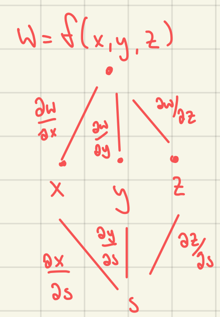
$\Rightarrow dw/dr = (\sin t)(-\sin t) + (\cos t)(\cos t) + (1)(1) = \cos(2t) + 1$

$\Rightarrow dw/dt \Big|_{t=0} = \cos(0) + 1 = 2$

<u>Example</u>: $\partial w/\partial r$ and $\partial w/\partial s$ in terms of $r$ and $s$

$w = x + 2y + z^2$ , $x = r/s$ , $y = r^2 + \ln s$ , $z = 2r$




We can write:

$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$

$\partial w/\partial s = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$

↓

| | | |
|---|---|---|
| $\partial w/\partial x = 1$ | $\partial x/\partial r = 1/s$ | $\partial x/\partial s = -r/s^2$ |
| $\partial w/\partial y = 2$ | $\partial y/\partial r = 2r$ | $\partial y/\partial s = 1/s$ |
| $\partial w/\partial z = 2z$ | $\partial z/\partial r = 2$ | $\partial z/\partial s = 0$ |

This all gives...

$$\partial w/\partial r = (1)(1/s) + (2)(2r) + (2z)(2)$$
$$\Rightarrow 1/s + 12r$$

Similarly...

$$\partial w/\partial s = (1)(-r/s^2) + (2)(1/s) + (2z)(0)$$
$$\Rightarrow 2/s - r/s^2$$

□ Differentiation of Implicitly defined curves and surfaces.

---

## Chain Rule

### Implicit Differentiation:

Suppose we have a curve defined implicitly; that is $F(x,y) = 0$

Taking $d/dx$ on both sides.

$$\Rightarrow d/dx\, F(x,y) = d/dx\,(0)$$

$$\Rightarrow \partial F/\partial x \; d/dx(x) + \partial F/\partial y \; d/dx(y) = 0$$

$$\boxed{\Rightarrow \frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}}$$

$$y^2 + y = x$$
$$\Rightarrow \frac{d}{dx}(y^2+y) = \frac{d}{dx}(x)$$
$$2y\, dy/dx + dy/dx = 1$$
$$dy/dx\,(2y+1) = 1$$
$$dy/dx = 1/(2y-1)$$

Example: $y^2 - x^2 - \sin(xy) = 0$

Let $F(x,y) = y^2 - x^2 - \sin(xy)$

The implicitly defined curve now becomes $F(x,y) = 0$

$$F_x = \partial F/\partial x = -2x - \cos(xy)(y)$$
$$F_y = \partial F/\partial y = 2y - \cos(xy)(x)$$

$$dy/dx = -F_x/F_y \Rightarrow dy/dx = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

$$d/dx\,(y^2 - x^2 - \sin(xy)) = 0$$

$$\Rightarrow 2y\, dy/dx - 2x - \cos(xy)\, d/dx\,(x \cdot y) = 0$$

$$\Rightarrow 2y\, dy/dx - 2x - \cos(xy)\left[y + x\, dy/dx\right] = 0$$

$$\Rightarrow dy/dx = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

## Implicitly Defined Surfaces:

Now suppose that we have an implicitly defined surface.

$F(x,y,z)=0$ , $z=f(x,y)$ (explicit form)

Taking the derivative on both sides w.r.t. $x$,

$\partial/\partial x\, F(x,y,z) = \partial/\partial x (0)$

$$\Rightarrow \partial F/\partial x \underbrace{\partial/\partial x (x)}_{=1} + \partial F/\partial y \cancel{\partial/\partial x (y)}^{0} + \partial F/\partial z \underbrace{\partial/\partial x (z)}_{\partial z/\partial x} = 0$$

$$\boxed{\Rightarrow \partial z/\partial x = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-F_x}{F_z}}$$

Provided that $F_z \neq 0$!

Similarly...

$$\boxed{\partial z/\partial y = \frac{-F_y}{F_z}}$$

Provided that $F_z \neq 0$!

Example $\partial z/\partial x$ and $\partial z/\partial y$ at $(0,0,0)$

$$x^3 + z^2 + ye^{xz} + z\cos y = 0 \longrightarrow ①$$

We take $F(x,y,z) = x^3 + z^2 + ye^{xz} + z\cos y$

$\Rightarrow \partial F/\partial x = 3x^2 + yze^{xz}$ ; $\partial F/\partial y = e^{xz} - z\sin y$ ; $\partial F/\partial z = 2z + xye^{xz} + \cos y$

$$\Rightarrow \partial z/\partial x \Big|_{(0,0,0)} = \frac{-F_x}{F_z}\Big|_{(0,0,0)} = 0/1 = 0$$

$$\Rightarrow \partial z/\partial y \Big|_{(0,0,0)} = \frac{-F_y}{F_z}\Big|_{(0,0,0)} = -1/1 = -1$$

# Quiz on 14.4

100%

✓ 1. $f(x,y) = x^2y^2$ w/ $x(t) = t^2$ and $y(t) = 1-t$

$df/dt \Big|_{t=-1}$

$\partial f/\partial x \cdot dx/dt + \partial f/\partial y \cdot dy/dt$

$(2xy^2)(2t) + (x^2 2y)(-1)$

$2(t^2)(1-t)^2(2t) + (t^2)^2 2(1-t)(-1)$

$6t^5 - 10t^4 + 4t^3$

$= -20$

✓ 2. $z = e^{xy}$ w/ $x = 2t + s$ and $y = t/s$

$dz/dt \Big|_{t=1,\, s=-2}$ $\quad \partial x/\partial t \quad \partial y/\partial t$

$(e^{xy}y)(\ 2\ ) + (e^{xy}x)(\ 1/s\ )$

$e^{(2t+s)(t/s)}(t/s)(2) + e^{(2t+s)(t/s)}(2t+s)(1/s)$

$= -1$

✓ 3. $\partial z/\partial y \quad (1,2,-2) \qquad x^2+y^2+z^2 = 9$

$\partial z/\partial y = -F_y/F_z$

$\partial F/\partial y = 2y \qquad = -2y/2z$

$\partial f/\partial z = 2z \qquad = -2(2)/2(-2)$

$= 1$

# 14.5 – Directional Derivatives

- ☐ Interpret the definition of directional derivatives
- ☐ Compute directional derivatives using the gradient vectors

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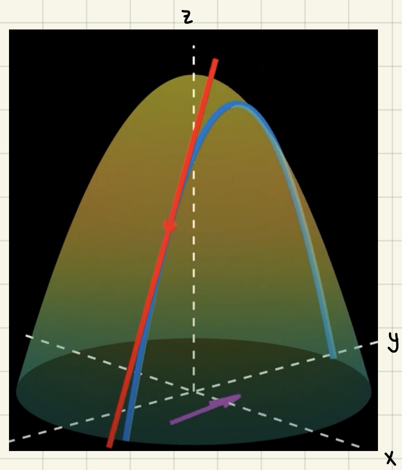
## Directional Derivatives

What is the slope in any direction?

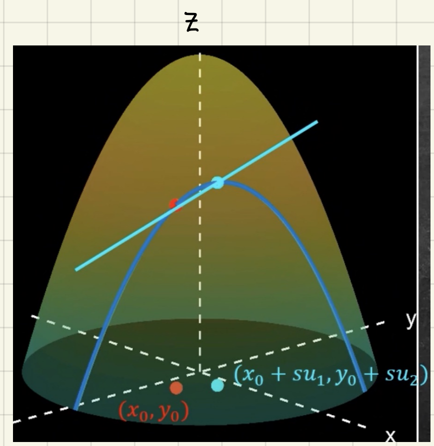



Fix a direction $\vec{u} = \langle u_1, u_2 \rangle$, where $|\vec{u}| = 1$.

$x(s) = x_0 + su_1$ ,

$y(s) = y_0 + su_2$

$$\boxed{D_{\vec{u}} f(x_0,y_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0,y_0)}{s}}$$




$$= \frac{d}{ds}\left[ f(x(s), y(s)) \right] \Big|_{s=0}$$

$$= \left( \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} \frac{dx}{ds} \right) + \left( \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \frac{dy}{dx} \right)$$

$$= \left( \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} u_1 \right) + \left( \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} u_2 \right)$$

Let $\nabla f = \langle \partial f/\partial x , \partial f/\partial y \rangle$, the "gradient of $f$"

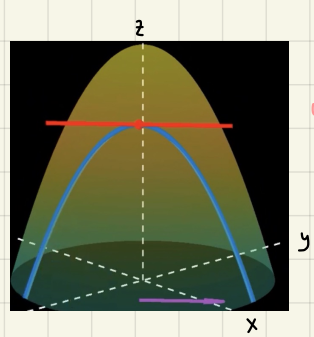
$$\boxed{D_{\vec{u}} f(x_0,y_0) = \nabla f|_{(x_0,y_0)} \cdot \vec{u}}$$

Example: $f(x,y) = 2 - x^2 - y^2$ @ $(1/2, -1/2)$
in direction $\vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$.

$D_{\vec{u}} f(x_0,y_0) = \langle -2x, -2y \rangle \big|_{(1/2,-1/2)} \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

$= \langle -1, 1 \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

$= 0$



- [ ] Interpret the geometric meaning of the gradient vector
- [ ] Describe the relationship between tangent and gradient vectors along a level curve.

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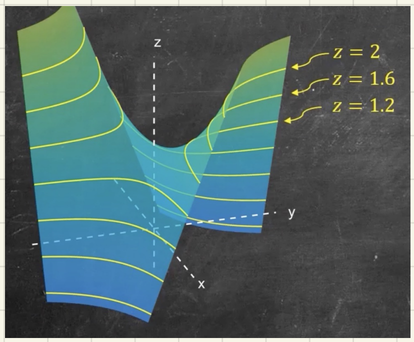
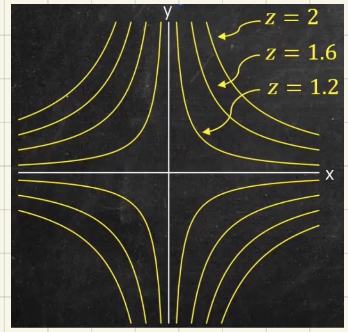
## Geometric Meaning of the Gradient




← Contour Curves (level curves)




Suppose along a curve $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$
That $f(x(t), y(t)) = C$.

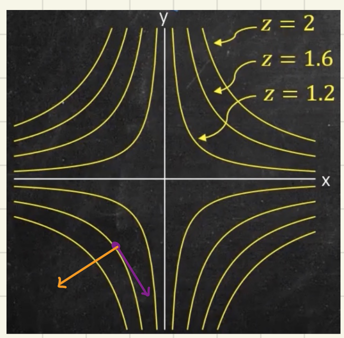
$$\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt} C$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}\right)\cdot\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}\right) = 0$$

$$\boxed{\nabla f \cdot \frac{d\vec{r}}{dt} = 0}$$

Normal •   Tangent •




Remember...

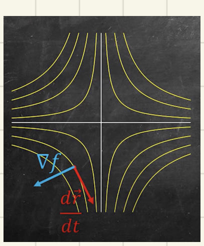
$$\boxed{\begin{aligned} D_{\vec{u}} f(x_0, y_0) &= \nabla f\big|_{(x_0, y_0)} \cdot \vec{u} \\ &= |\nabla f||\vec{u}|\cos\theta \end{aligned}}$$

- Smallest magnitude zero when $\theta = \pi/2$.

  i.e. $d\vec{r}/dt$ is direction of minimum slope.

- Largest magnitude one when $\theta = 0$.

  i.e. $\nabla f$ is direction of maximum slope.

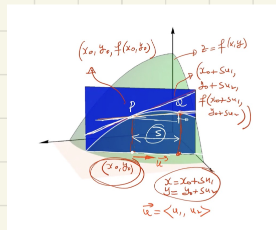


- ☐ Definition of Directional Derivative (limit and using the gradient)
- ☐ Directions of most rapid change
- ☐ Tangent lines to level curves

---

## Directional Derivatives and Gradient Vectors

The derivative of $f$ at $P(x_0, y_0)$ in the direction of the unit vector $\vec{u} = \langle u_1, u_2 \rangle$ is the number...



$z = f(x,y)$

$x = x_0 + su_1$

$y = y_0 + su_2$

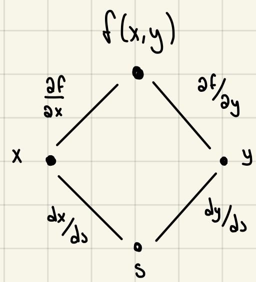
$\vec{u} = \langle u_1, u_2 \rangle$

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

Provided the limit exists.

So...

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$




Where $z = f(x,y)$ ; $x = x_0 + su_1$ ; $y = y_0 + su_2$

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle$$

"gradient of $f$"

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle u_1, u_2 \rangle$$

$$= \vec{\nabla} f \cdot \vec{u} \quad \text{where } \vec{\nabla} f = \langle f_x, f_y \rangle$$

$$\boxed{D_{\vec{u}} f \Big|_P = \vec{\nabla} f \Big|_P \cdot \vec{u}}$$

We have $(D_{\vec{u}} f)\big|_P = \vec{\nabla} f \big|_P \cdot \vec{u}$

$$= \left|\vec{\nabla} f\big|_P\right| \left|\vec{u}\right| \cos\theta$$

$$= \left|\vec{\nabla} f\big|_P\right| \cos\theta$$

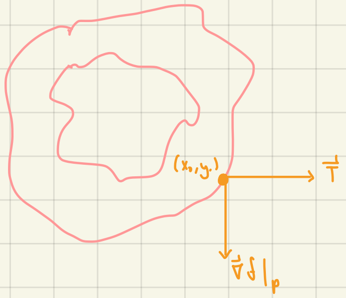
① At each point in the domain of $f$, the maximum value of the directional derivative comes when $\cos\theta = 1$.
$f$ increases most rapidly in the direction of $\vec{\nabla} f$ at $P$.

② $f$ decreases most rapidly in the direction $-\vec{\nabla} f$.

③ Any direction $\vec{u}$ orthogonal to $\vec{\nabla} f$ is the direction of zero change. $(\vec{\nabla} f|_p \neq 0.)$

Suppose we have a smooth curve $\vec{r}(t) = \langle x(t), y(t) \rangle$, s.t

$f(x(t), y(t)) = C$ ; $z = f(x,y)$




$\Rightarrow \frac{d}{dt} f(x(t), y(t)) = 0$

$\vec{\nabla} f = \langle f_x, f_y \rangle$

$\vec{T} = \langle -f_y, f_x \rangle$

$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0$

$x = x_0 + t(-f_y)$
$y = y_0 + t(f_x)$

$\Rightarrow \underbrace{\vec{\nabla} f} \cdot \underbrace{d\vec{r}/dt}_{\vec{T}} = 0$

$\Rightarrow \frac{x - x_0}{-f_y} = \frac{y - y_0}{f_x}$ $\boxed{\Rightarrow (x - x_0) f_x + (y - y_0) f_y = 0}$

Equation of tangent line

<u>Summary</u>

- $(D_{\vec{u}} f)_{p(x_0,y_0)} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$ ; $\vec{u} = \langle u_1, u_2 \rangle$ and $|\vec{u}| = 1$.
- $(D_{\vec{u}} f)\big|_p = (\vec{\nabla} f)\big|_p \cdot \vec{u}$ ; $\vec{\nabla} f = \langle \partial f/\partial x, \partial f/\partial y \rangle$
- $f$ increases (decreases) most rapidly in the direction of $\vec{\nabla} f|_p$ $(-\vec{\nabla} f|_p)$
- $(x - x_0) f_x \big|_{p(x_0,y_0)} + (y - y_0) f_y \big|_{p(x_0,y_0)} = 0$ (Equation of tangent line)
- $\frac{d}{dt} f(\vec{r}(t)) = \vec{\nabla} f(\vec{r}(t)) \cdot d\vec{r}/dt$

- Examples on:
  - Directional Derivatives (by limit and using the gradient)
  - Directions of most rapid increase and decrease, and zero change
  - Finding tangents on leve curves

---

## Directional Derivatives

↓

Example: using th definition, find th derivative of

$$f(x,y) = x^2 + xy$$

at $P_0(1,2)$ in th direction of th unit vector

$$\vec{u} = \langle \overset{u_1}{1/\sqrt{2}}, \overset{u_2}{1/\sqrt{2}} \rangle$$

Remember: $$(D_{\vec{u}} f)_{p(x_0,y_0)} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

$$= \lim_{s\to 0} \frac{(1+ s/\sqrt{2})^2 + (1+ s/\sqrt{2})(2+ s/\sqrt{2}) - [1^2 + (1)(2)]}{s}$$

$$= \lim_{s\to 0} \frac{(\cancel{1} + s^2/2 + 2s/\sqrt{2}) + (\cancel{2} + s^2/2 + 3s/\sqrt{2}) - \cancel{3}}{s}$$

$$= \lim_{s\to 0} (s + 5/\sqrt{2})$$

$$(D_{\vec{u}} f)\Big|_{(1,2)} = 5/\sqrt{2}$$

Example: Find th derivative of $f(x,y) = xe^y + \cos(xy)$ at th point $(2,0)$ in th direction of $\vec{v} = \langle 3, -4 \rangle$.

$$(D_{\vec{u}} f)\Big|_p = (\vec{\nabla} f)\Big|_p \cdot \vec{u}$$

For $\vec{v} = \langle 3, -4 \rangle$
$\vec{u} = \vec{v}/|\vec{v}| = \langle 3/5, -4/5 \rangle$
$|\vec{v}| = 5.$

$(x_0, y_0) = (2,0)$, $\vec{u} = \langle 3/5, -4/5 \rangle$

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle e^y - \sin(xy)(y), xe^y - \sin(xy)(x) \right\rangle$$

$$\Rightarrow \vec{\nabla} f \Big|_{p=(2,0)} = \langle 1-0, 2 \rangle = \langle 1, 2 \rangle$$

$$\Rightarrow (D_{\vec{u}} f)\Big|_{(2,0)} = (\vec{\nabla} f)\Big|_p \cdot \vec{u} = \langle 1,2 \rangle \cdot \langle 3/5, -4/5 \rangle = -1$$

Example: Find th direction in which $f(x,y) = (x^2/2) + (y^2/2)$

a) increases most rapidly at th point $(1,1)$, and
b) decreases most rapidly at th point $(1,1)$.
c) What are th directions of zero change in $f$ at $(1,1)$.

↓

a) $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle x, y \rangle$ ; $\Rightarrow \vec{\nabla} f \big|_{(1,1)} = \langle 1,1 \rangle$

$\vec{u} = \dfrac{\langle 1,1 \rangle}{|\langle 1,1 \rangle|} = \left\langle 1/\sqrt{2}, 1/\sqrt{2} \right\rangle$ is th direction in which $f$ increases most rapidly.

b) $f$ decreases most rapidly in th direction $\langle -1/\sqrt{2}, -1/\sqrt{2} \rangle$.

c) Direction of zero change is th direction orthogonal to $(\vec{\nabla} f)\big|_p$

$\vec{n}_1 = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$ , $\vec{\nabla} f \big|_p = \langle 1,1 \rangle$

and

$\vec{n}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

Example: find an equation for th tangent to th ellipse $x^2/4 + y^2 = 2$ at th point $(-2,1)$.

We have $(x - x_0) f_x \big|_p + (y - y_0) f_y \big|_p = 0$

Consider $f(x,y) = x^2/4 + y^2 - 2 \Rightarrow f(x,y) = 0 \Rightarrow x^2/4 + y^2 = 2$

$f_x = \dfrac{\partial f}{\partial x} = x/2$ , $f_y = \dfrac{\partial f}{\partial y} = 2y$

$\partial f/\partial x \big|_p = f_x \big|_p = -1$ , $\partial f/\partial y \big|_p = f_y \big|_p = 2$

With $x_0 = -2$, $y_0 = 1$ ; $(x+2)(-1) + (y-1)(2) = 0$

$-x - 2 + 2y - 2 = 0$ $\boxed{\Rightarrow -x + 2y = 4}$

Example: a) find th derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in direction of $\vec{v} = \langle 2,-3,6 \rangle$.
b) in what direction does $f$ change most rapidly at $P_0$, and what are th rates of change in those directions?

$\vec{\nabla} f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle$

a) $\vec{u} = \vec{v}/|\vec{v}| = 1/7 \langle 2,-3,6 \rangle$

$\vec{\nabla} f = \langle 3x^2 - y^2, -2xy, -1 \rangle \Rightarrow \vec{\nabla} f \big|_p = \langle 2, -2, -1 \rangle$

$\Rightarrow (D_{\vec{v}} f)\big|_p = \langle 2,-2,-1 \rangle \cdot 1/7 \langle 2,-3,6 \rangle$

$= 4/7 + 6/7 - 6/7 = \boxed{4/7}$

$\downarrow$

b) The direction in which $f$ increases most rapidly is:

$$\frac{\vec{\nabla} f\big|_p}{\left|\vec{\nabla} f\big|_p\right|}$$

The direction of most rapid decreases is:

$$-\frac{\vec{\nabla} f\big|_p}{\left|\vec{\nabla} f\big|_p\right|}$$

$$(D_{\vec{u}} f)\big|_p = \vec{\nabla} f\big|_p \cdot \vec{u}$$

$$= \left|\vec{\nabla} f\big|_p\right| \cos\phi$$

In the two cases...

$$(D_{\vec{u}} f)\big|_p = \pm\left|\vec{\nabla} f\big|_p\right|$$

The rate of change in th above directions are, respectively:

$$\left|\vec{\nabla} f\big|_p\right| \text{ and } -\left|\vec{\nabla} f\big|_p\right|$$

# Quiz on 14.5

100%

✓ 1. $f(x,y) = xe^y$ at $(3,0)$ $\vec{v} = \langle -4, 3 \rangle$

$\vec{u} = \langle -4/5, 3/5 \rangle$

$\langle e^y, xe^y \rangle$

$= \langle 1, 3 \rangle$

$\langle 1, 3 \rangle \cdot \langle -4/5, 3/5 \rangle$

$= 1$

✓ 2. $h(x,y,z) = \cos(xy) + e^{yz} + \ln(zx)$ at $P_0 = (1, 0, 1/2)$ $\vec{v} = \langle 1, 2, 2 \rangle$

$\vec{u} = \langle 1/3, 2/3, 2/3 \rangle$

$\vec{\nabla} f = \langle 1/x - y\sin(xy), e^{yz}z - x\sin(xy), e^{yz}y + 1/z \rangle$

$\langle 1, 1/2, 2 \rangle$

$= 2$

✓ 3. Directional Derivative $f(x,y,z) = xe^y + z^2$, $f(x,y,z)$ increases at $P_0 = (1, \ln(2), 1/2)$

$\vec{\nabla} f = \langle e^y, e^y x, 2z \rangle$

$= \langle 2, 2, 1 \rangle$

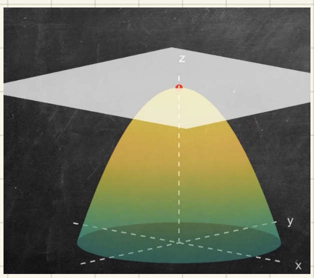
$= \sqrt{2^2 + 2^2 + 1^2}$

$= 3$

# 14.6 — Tangent Planes and Linearization

- [ ] Compute the equation of a tangent plane at a point on a surface of the form $z = f(x,y)$.

## Equation of Tangent Planes

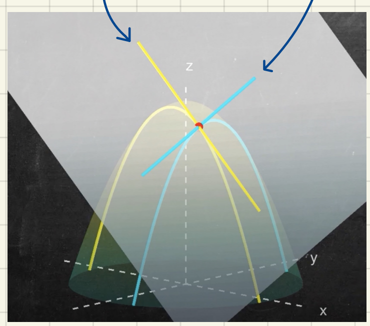



Goal: A plane that meets at $z_0 = f(x_0, y_0)$ and is "close" to $z = f(x,y)$ "near by".

$$\boxed{\text{Equation of Plane: } \vec{n} \cdot \overrightarrow{P_0P} = 0}$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\boxed{\therefore\ f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0 = z}$$




Example

Choose $c = -1$

$$\underbrace{a(x-x_0) + b(y-y_0) + z_0}_{\text{A linear function } L(x,y)} = z$$

Plug in $y = y_0$

$$\underbrace{a(x-x_0) + z_0}_{\text{The equation of a line}} = z$$

Linear approximation with $a = f_x(x_0, y_0)$

Plug in $x = x_0$

$$b(y-y_0) + z_0 = z$$

Linear approximation with $b = f_y(x_0, y_0)$

Example $f(x,y) = 2 - x^2 - y^2$ @ $(\frac{1}{2}, \frac{1}{2})$

$$z_0 = f(\tfrac{1}{2}, \tfrac{1}{2}) = 2 - (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 = 3/2$$

$$f_x(\tfrac{1}{2}, \tfrac{1}{2}) = -2(\tfrac{1}{2}) = -1$$

$$f_y(\tfrac{1}{2}, \tfrac{1}{2}) = -2(\tfrac{1}{2}) = -1$$

$$\boxed{z = 3/2 - 1(x - \tfrac{1}{2}) - 1(y - \tfrac{1}{2})}$$

Equation of the tangent line!

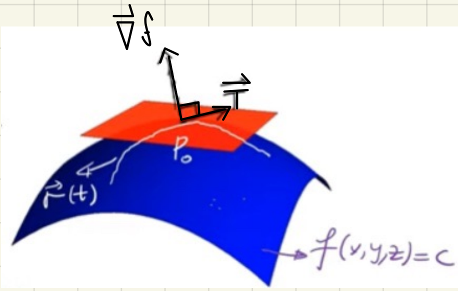
- ☐ Equation of the Tangent Plane
- ☐ Equation of the Normal Line
- ☐ Linearization of $z = f(x,y)$ using the Tangent Plane
- ☐ Total differential

---

## Main Formulas

### Equation of a Tangent Plane

Suppose $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a smooth curve on the level surface $f(x,y,z) = c$, and that $f(x,y,z)$ is differentiable function. $P_0 = (x_0, y_0, z_0)$ is a point on the tangent plane.




$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{d}{dt}(c)$$

$$\vec{\nabla} f \big|_{P} \cdot \underbrace{\frac{d\vec{r}}{dt}}_{\vec{T}} = 0$$

That means that $\vec{\nabla} f \big|_{P_0}$ can be used as a normal vector to the tangent plane i.e.,

$$\vec{n} = \langle f_x \big|_{P_0}, f_y \big|_{P_0}, f_z \big|_{P_0} \rangle$$

The tangent plane at the point $P_0 = (x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ is then given by,

$$f_x \big|_{P_0} (x - x_0) + f_y \big|_{P_0} (y - y_0) + f_z \big|_{P_0} (z - z_0) = 0$$

Remember $\vec{n} \cdot \overrightarrow{P_0P} = 0$

If $z = f(x,y) \Rightarrow f(x,y) - z = 0$, In this case:

$$z = z_0 + f_x \big|_{P_0} (x - x_0) + f_y \big|_{P_0} (y - y_0)$$

where $z_0 = f(x_0, y_0)$

### Equation of the Normal Line

The equation of the normal line passing through the point $P_0 = (x_0, y_0, z_0)$ in the direction of $\vec{\nabla} f \big|_{P_0}$ is given by,

$$x = x_0 + f_x \big|_{P_0} t$$
$$y = y_0 + f_y \big|_{P_0} t$$
$$z = z_0 + f_z \big|_{P_0} t$$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

## Linearization

The equation of the tangent line is given by,

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a) = \mathcal{L}(x)$$

The equation of the tangent plane is given by,

$$z = f(x_0, y_0) + f_x\big|_{P_0}(x-x_0) + f_y\big|_{P_0}(y-y_0)$$

The linearization of $f(x,y)$ at $P_0$ is:

$$\boxed{\mathcal{L}(x,y) = f(x-x_0) + f_x\big|_{P_0}(x-x_0) + f_y\big|_{P_0}(y-y_0)}$$

## Total Differential

Remember: $\dfrac{dy}{dx} = f'(x)\Big|_{x=a} \longrightarrow dy = f'(a)\,dx$

If we move from $(x_0, y_0)$ to a point $\overset{(x_0+dx,\ y_0+dy)}{(x_0+\Delta x,\ y_0+\Delta y)}$, then...

$$df = f_x\big|_{(x_0,y_0)}dx + f_y\big|_{(x_0,y_0)}dy$$

is called the <u>total differential</u> of $f$.

## Summary

<u>Equation of tangent plane</u>: $f_x\big|_{P_0}(x-x_0) + f_y\big|_{P_0}(y-y_0) + f_z\big|_{P_0}(z-z_0) = 0$

$z = f(x,y)$: $z = f(x_0, y_0) + f_x\big|_{P_0}(x-x_0) + f_y\big|_{P_0}(y-y_0)$

<u>Normal line</u>: $x(t) = x_0 + f_x\big|_{P_0}\cdot t$, $y(t) = y_0 + f_y\big|_{P_0}\cdot t$, $z(t) = z_0 + f_z\big|_{P_0}\cdot t$

<u>Linearization</u>: $\mathcal{L}(x,y) = f(x_0,y_0) + f_x\big|_{P_0}(x-x_0) + f_y\big|_{P_0}(y-y_0)$

$$\mathcal{L}(x,y) \approx f(x,y)$$

<u>Total differential</u>: $df = f_x\big|_{P_0}dx + f_y\big|_{P_0}dy$

$$L(x_0+\Delta x, y_0+\Delta y) - L(x_0, y_0)$$

- ☐ Computing examples on:
  - ☐ Equation of the Tangent Plane
  - ☐ Equation of the Normal Line
  - ☐ Linearization of $z = f(x,y)$ using the Tangent Plane
  - ☐ Total differential

---

## Examples

**Example** Tangent plane and Normal line of the level surface ...

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0 \quad @ \ (1,2,4)$$

We have the equation of a tangent plane: $f_x|_{P_0}(x-x_0) + f_y|_{P_0}(y-y_0) + f_z|_{P_0}(z-z_0) = 0$

$$\vec{\nabla} f = \langle 2x, 2y, 1 \rangle \Rightarrow \vec{\nabla} f|_{P_0} = \langle 2, 4, 1 \rangle$$

$\Rightarrow$ The equation of the plane is...

$$2(x-1) + 4(y-2) + 1(z-4) = 0$$

$$\boxed{\Rightarrow 2x + 4y + z = 14}$$

The equation of the normal line at $P_0 = (1,2,4)$ is:

$$\begin{aligned} x &= x_0 + f_x|_{P_0} t \\ y &= y_0 + f_y|_{P_0} t \\ z &= z_0 + f_z|_{P_0} t \end{aligned}$$

this gives:

$$\boxed{\begin{aligned} x &= 1 + 2t \\ y &= 2 + 4t \\ z &= 4 + t \end{aligned}}$$

**Example** plane tangent to the surface $z = x\cos y - ye^x$ @ $(0,0,0)$

Let $g(x,y,z) = x\cos y - ye^x - z = 0$

$g_x|_{P_0}(x-x_0) + g_y|_{P_0}(y-y_0) + g_z|_{P_0}(z-z_0) = 0$ OR $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$$f(0,0) = 0$$

↓

$$\frac{\partial f}{\partial x} = \cos y - ye^x \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 1$$

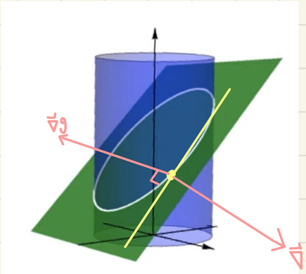
$$\frac{\partial f}{\partial y} = -x\sin y - e^x \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = -1$$

Required equation is...

$$z = 0 + 1(x-0) + (-1)(y-0) \Rightarrow \boxed{x - y - z = 0}$$

Example The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ AND $g(x,y,z) = x + z - 4 = 0$ meet in an ellipse. find the parametric equations for the line tangent to this ellipse @ $(1,1,3)$

The direction vector of the tangent line can be found by doing...




$$\vec{v} = \vec{\nabla} f\big|_{(1,1,3)} \times \vec{\nabla} g\big|_{(1,1,3)} = \langle 2x, 2y, 0\rangle\big|_{(1,1,3)} \times \langle 1,0,1\rangle\big|_{(1,1,3)}$$
$$= \langle 2, -2, -2\rangle$$

The parametric equations of the tangent lines at $(1,1,3)$, with direction vector $\vec{v} = \langle 2,-2,-2\rangle$ is then given by:

$$\boxed{x = 1 + 2t \;;\; y = 1 - 2t \;;\; z = 3 - 2t}$$

Example Find the linearization of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ @ $(3,2)$.

The linearization $L(x,y)$ for a surface $z = f(x,y)$ at $(x_0, y_0)$ is ...

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$f(3,2) = 3^2 - (3)(2) + \tfrac{1}{2}(2)^2 + 3$$
$$= 8$$

$$\partial f/\partial x = 2x - y \Rightarrow f_x\big|_{(3,2)} = 2(3) - 2 = 4$$

$$\partial f/\partial y = -x + y \Rightarrow f_y\big|_{(3,2)} = -3 + 2 = -1$$

$$\Rightarrow L(x,y) = 8 + 4(x-3) + (-1)(y-2)$$

$$\Rightarrow L(x,y) \boxed{= 4x - y - 2}$$

$$f(x,y) \approx L(x,y)$$

↓

**Example** find the linearization $L(x,y,z)$ of $f(x,y,z) = x^2 - xy + 3\sin z$ @ $(2,1,0)$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

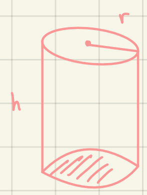
$$f(2,1,0) = 2^2 - (2)(1) + 0 = 2$$

$$f_x = 2x - y \Rightarrow f_x(2,1,0) = 3 \;;\; f_y = -x \Rightarrow f_y(2,1,0) = -2 \;;\; f_z = 3\cos z \Rightarrow f_z(2,1,0) = 3$$

$$\Rightarrow L(x,y,z) = 2 + 3(x-2) + (-2)(y-1) + 3(z-0)$$

$$\boxed{= 3x - 2y + 3z - 2}$$

**Example** Estimate the change : cylinder w/ $r=1$, $h=5$ $dr = +0.03$ and $dh = -0.1$
Estimate the resulting absolute change in the volume of the can.




$V = \pi r^2 \cdot h$

$$df = f_x\big|_{P_0} dx + f_y\big|_{P_0} dy$$

$$\Delta V \approx dv = V_r(1,5)\,dr + V_h(1,5)\,dh$$

$$V_r = 2\pi r h \;,\; V_h = \pi r^2$$

$$\Delta V \approx (2\pi r h)\big|_{(1,5)}(0.03) + (\pi r^2)\big|_{(1,5)}(-0.1)$$

$$= 0.2\pi$$
$$\approx 0.63 \text{ in}^3$$

**Example** Estimate the change : Cylindrical $h=25$ and $r=5$.
How sensitive are the tanks' volumes to small variations in height and radius?

$V = \pi r^2 \cdot h$
$V_r = 2\pi r h$
$V_h = \pi r^2$

$$dv = V_r(5,25)\,dr + V_h(5,25)\,dh$$

$$= (2\pi r h)\big|_{(5,25)} dr + (\pi r^2)\big|_{(5,25)} dh$$

$$\boxed{= 250\pi\,dr + 25\pi\,dh}$$

$\underline{dr = 0}$ $\quad dv = 25\pi\,dh$ $\quad$ change in h by 1 unit, $dh = 1$, changes volume by $25\pi$

$\underline{dh = 0}$ $\quad dv = 250\pi\,dr$ $\quad$ Variation of 1 in dr gives $dv = 250\pi$

Volume is 10 times more sensitive in this case!

# Quiz on 14.6

~~0.8/1~~ 0.8/1 $\boxed{2.8/3}$

1. Tangent plane to the surface $z = f(x,y) = x^3 - y^2$ @ $(1,2)$

$\vec{\nabla} f = \langle 3x^2, -2y \rangle$

$= \langle 3, -4 \rangle$

$2x - 2 - 4y + 4 = 0$

$2x - 4y + 2$

?!

$2(x-1) + (-4)(y-2) = 0$

+1 $z_0 = f(x_0, y_0)$

$= x^3 - y^2$

$= (1)^3 - (2)^2$

$= -3$

2. $L(x,y)$ $f(x,y) = x^3y^4$ @ $(1,1)$

$f(1,1) = 1$

$\partial f/\partial x = 3x^2y^4 = 3$

$\partial f/\partial y = 4x^3y^3 = 4$

$L(x,y) = 1 + 3(x-1) + 4(y-1)$

$= 3x - 3 + 4y - 4 + 1$

$= 3x + 4y - 6$

3. Parametric Equations of Tangent line 0.67/1

$x + y^2 + z = 2$ and $y = 1$ @ $(1/2, 1, 1/2)$

$\langle 1, 2y, 1 \rangle$ $\langle 0, 1, 0 \rangle$

$\langle 1, 2, 1 \rangle \times \langle 0, 1, 0 \rangle$

$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\imath}(-1) + \hat{\jmath}(0-0) + \hat{k}(1)$

$= -1\hat{\imath}, 0\hat{\jmath}, 1\hat{k}$

$x = 1/2 + -1t$ ← BS "Assume coefficient is negative"

$y = 1 + 0t$

$z = 1/2 + 1t$

# 14.7 – Optimization

☐ Main theorems on finding maximums and minimums for a function of two variables

---

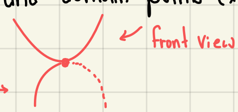
## Optimization and the Second Derivative Test ( Extreme Values and Saddle Points )

△ Let $f(x,y)$ be defined on a region $R$ containing a point $(a,b)$, then

- $f(a,b)$ is a local maximum value of $f$ if $f(a,b) \geq f(x,y)$ for all domain points $(x,y)$ in a open disk centered at $(a,b)$.
- $f(a,b)$ is a local minimum value of $f$ if $f(a,b) \leq f(x,y)$ for all domain points $(x,y)$ in an open disk centred at $(a,b)$.

△ An interior point of the domain of $f(x,y)$ where both $f_x$ and $f_y$ are zero or at least one of $f_x$ of $f_y$ do not exist is a critical point of $f$.

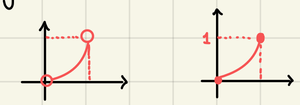
△ A differentiable function $f(x,y)$ has a saddle point at a critical point $(a,b)$ if in every open disk centred at $(a,b)$ there are some domain points $(x,y)$ where $f(x,y) > f(a,b)$ and domain points $(x,y)$ where $f(x,y) < f(a,b)$. The point $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is a saddle point.




☐ Extreme values of $f(x,y)$:

A function $z=f(x,y)$ that is continuous on a closed, bounded set $R$ in the plane. The function takes on an absolute maximum and an absolute minimum at some point in $R$.

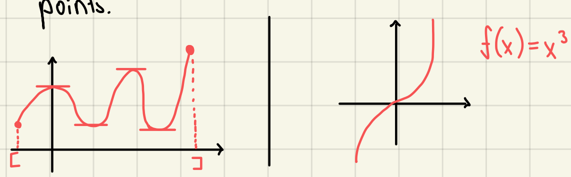
ex. $y=x^2$: $(0,1)$ , $[0,1]$




**Theorem:** If $f(x,y)$ has a local maximum or minimum value at an interior point $(a,b)$ of its domain and if partial derivatives (first-order) exist at $(a,b)$, then...

$\boxed{f_x(a,b)=0}$ and $\boxed{f_y(a,b)=0}$

Note: This tells us that the only points where $f(x,y)$ can take extreme values are critical points and boundary points.




**Theorem:** Second derivative test for local extreme values.

Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at $(a,b)$ and that...

$f_x(a,b)=0$, $f_y(a,b)=0$, then ...  Note: $\boxed{D = f_{xx}f_{yy} - f_{xy}^2}$

① $f$ has a local maximum at $(a,b)$ if $f_{xx} < 0$ and $D > 0$ at $(a,b)$.
② $f$ has a local minimum at $(a,b)$ if $f_{xx} > 0$ and $D > 0$ at $(a,b)$.

3. $f$ has a saddle point at $(a,b)$ if $D<0$ at $(a,b)$.
4. If $D=0$ at $(a,b)$, the test is Inconclusive.

☐ Computing examples on finding maximums and minimums for a function of two variables

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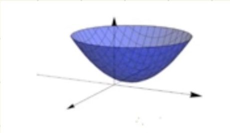
## Optimization and the Second Derivative Test

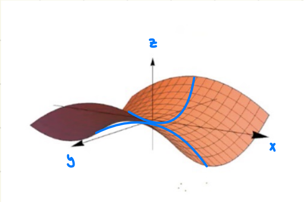
**Example 1**: Find the local extreme values of $f(x,y) = x^2+y^2-4y+9$



Here $f_x = 2x = 0$ and $f_y = 2y-4=0$
$\Rightarrow x=0$ and $\Rightarrow y=2$

$(0,2)$ is a critical point.
$(0,2)$ is the local minimum for $f(x,y)$.

**Example 2**: find the local extreme values (if any) of $f(x,y) = y^2-x^2$




Setting $f_x=0$, $f_y=0$

$\Rightarrow -2x=0$, $2y=0$
$(0,0)$ is the only critical point.

$\therefore$ $(0,0)$ is indeed a saddle point.

Surface in the xz-plane: $y=0$

$\Rightarrow z=-x^2$

Surface in the yz-plane: $x=0$

$\Rightarrow z=y^2$

**Example 3**: Find the local extreme values of the function $f(x,y) = xy-x^2-y^2-2x-2y+4$

$f_x = y-2x-2$, $f_y = x-2y-2$
Set $f_x=0$, $f_y=0$
$-2x+y=2$, $x-2y=2$

$\Rightarrow -4x+x = 4+2$
$\Rightarrow x=-2$

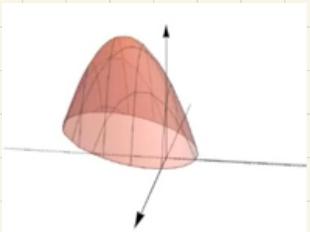
also, we get $y=-2$

$(-2,-2)$ is the only critical point here.

$f_{xx} = -2 < 0$, $f_{yy}=-2$, $f_{xy}=1$

$D = f_{xx}f_{yy} - f_{xy}^2$
$= (-2)(-2) - (1)^2$
$= 3 > 0$

Since $f_{xx}\big|_{(-2,-2)} = -2 < 0$
and $D\big|_{(-2,-2)} > 0$,
$(-2,-2)$ must be a local maximum.



**Example 4:** Find the local extreme values of the function $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

$f_x = -6x + 6y$ , $f_y = 6y - 6y^2 + 6x$

Set $f_x = 0$ and $f_y = 0$

$\Rightarrow -6(x-y) = 0$ and $6(y - y^2 + x) = 0$

or $x - y = 0 \to ①$ and $y - y^2 + x = 0 \to ②$

take $x = y$ in eq. 2 $\Rightarrow x - x^2 + x = 0$

$\Rightarrow 2x - x^2 = 0$

$\Rightarrow x = 0,\ x = 2$

$\Rightarrow (0,0)$ and $(2,2)$ are the critical points.

$f_{xx} = -6 < 0$, $f_{yy} = 6 - 12y = 6(1-2y)$, $f_{xy} = 6$

$D = f_{xx} f_{yy} - f_{xy}^2$

$= (-6)(6 - 12y) - 6^2$

$= -36 + 72y - 36$

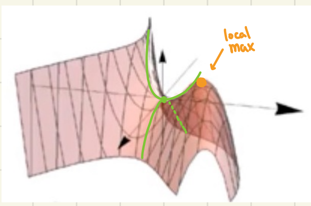
$= -72(1-y)$

$(0,0)$ : $D(0,0) < 0$

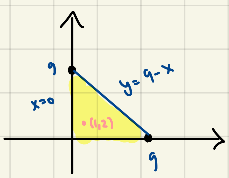
$(0,0)$ is a saddle point

$(2,2)$ : $D(2,2) = 72 > 0$

$(2,2)$ is a local maximum.




**Example 6:** Find the absolute maximum and minimum values of $f(x,y) = 2 + 2x + 4y - x^2 - y^2$ on the triangle region in the first quadrant bounded by the lines $x=0$, $y=0$, and $y = 9-x$.




$f_x = 0$ , $f_y = 0$

$\Rightarrow 2 - 2x = 0$ , $4 - 2y = 0$

$x = 1$ , $y = 2$

$(1,2)$ is a critical value.

$f(1,2) = 2 + 2 + 8 - 1 - 4 = 7$

Lets do the work on the boundary.

1) $y = 0$ , $0 \le x \le 9$

$f(x,0) = 2 + 2x - x^2$

$\frac{d}{dx} f(x,0) = 0 \iff x = 1$

$f(0,0) = 2$

$f(1,0) = 3$

$f(9,0) = -61$

2) $x = 0$ , $0 \le y \le 9$

$f(0,y) = 2 + 4y - y^2$

$\frac{d}{dy} f(0,y) = 0 \iff 4 - 2y = 0 \iff y = 2$

$f(0,0) = 2$

$f(0,2) = 6$

$f(0,9) = -43$

3) $y = 9 - x$

$f(x,y) = 2 + 2x + 4y - x^2 - y^2$

$f(x, 9-x) = 2 + 2x + 4(9-x) - x^2 - (9-x)^2$

$= -43 + 16x - 2x^2$

$\frac{d}{dx} f(x, 9-x) = 0 \iff 16 - 4x = 0 \iff x = 4$

then $y = 9 - x \Rightarrow y = 9 - 4 = 5$

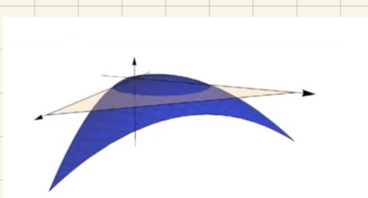
The point is $(4,5)$

$f(4,5) = -11$

$f(9,0) \to$ smallest

$f(1,2) \to$ largest

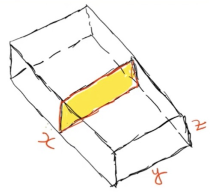
$f(x,y)$ has absolute maximum at $(1,2)$ and absolute minimum at $(9,0)$.



# Example 7

**Example-7:** Extreme values

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.




The volume of the box is given by

$$V(x,y,z) = x \cdot y \cdot z$$

We are given that $x + 2y + 2z = 108 \longrightarrow ①$

$\Rightarrow x = 108 - 2y - 2z$

$\Rightarrow V(y,z) = (108-2y-2z)(y)(z)$

or $V(y,z) = 108yz - 2y^2z - 2yz^2$

$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z)$

$V_z = y(108 - 2y - 4z)$

Set $V_y = 0$ and $V_z = 0$

$\Rightarrow z(108 - 4y - 2z) = 0 \longrightarrow ②$

$\Rightarrow y(108 - 2y - 4z) = 0 \longrightarrow ③$

Equation ② gives: $z = 0$ or $108 - 4y - 2z = 0$

For $z=0$, equation ③ gives $y(108-2y) = 0 \Rightarrow y=0,\ y = 54$

$(0,0)$ and $(54,0)$ are two critical points.

Equation ③ gives: $y=0$ or $108 - 2y - 4z = 0$

for $y=0$, equation ② gives: $z=0,\ z=54$

$(0,0)$ and $(0,54)$ are two more critical points.

These points are <u>NOT</u> feasible.

Now we look at ...

$108 - 4y - 2z = 0$

and $108 - 2y - 4z = 0$

These two have th solution

$y = 18,\ z = 18$ ; feasible.

$V_{yy} = -4z$

$V_{zz} = -4y$

$V_{yz} = 108 - 4y - 4z$

$V_{yy}\big|_{(18,18)} = -72 < 0$

$V_{zz}\big|_{(18,18)} = -72 < 0$

$V_{yz}\big|_{(18,18)} = -36$

$D = (V_{yy}V_{zz} - V_{yz})\big|_{(18,18)}$

$D(18,18) = (-72)(-72) - (-36)^2 > 0$

$V_{yy}(18,18) < 0,\ D(18,18) > 0$

So, $(18,18)$ is a maximum.

$x = 108 - 2y - 2z$

$\Rightarrow x = 36$ inches

<u>Dimensions</u>: $x = 36$ inches

$y = 18$ inches

$z = 18$ inches.

# 14.7 - Supplemental Videos

Quiz on 14.7

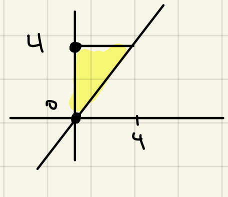
100%

✓ 1. $f(x,y) = x^2 + y^2 - pxy$, where $p$ is a constant
(0,0) is critical point

$f_{xx} = 2$ $f_{xy} = -p$
$f_{yy} = 2$

$D(0,0) = (2)(2) - (-p)^2$
$= 4 - p^2$
$\boxed{p = 2}$ $p = -2$

✓ 2 Abs. max $g(x,y) = x^2 - xy + y^2 + 1$




$x = 0,\ y = 4,\ y = x$

$f_x = 0 = 2x - y$
$2x = y$

$f_y = 0 = 2y - x$
$2y = x$

$2(2x) - x = 0$
$4x - x = 0$
$3x = 0$
$x = 0$

(0,0) critical point

$\boxed{g(0,0) = 1}$

$x = 0$ $0 \le y \le 4$
$f(0,y) = y^2 + 1$
$\frac{d}{dy} f(0,y) = 0 = 2y \iff y = 0$

$\boxed{\begin{array}{l} f(0,0) = 1 \\ f(0,0) = 1 \\ f(0,4) = 17 \end{array}}$

$x = y$
$f(x,y)$
$f(x,x) = x^2 - x^2 + x^2 + 1$
$x^2 + 1$
$= 2x$
$x = 0$
$x = y$
$0 = y$

$y = 0$ $0 \le x \le 4$
$f(x,0) = x^2 + 1$
$\frac{d}{dx} f(x,0) = 0 = 2x \iff x = 0$

$f(0,0) = 1$

$\boxed{\begin{array}{l} f(0,0) = 1 \\ f(0,0) = 1 \\ f(4,0) = 17 \end{array}}$

max = 17

✓ 3. Abs min $g(x,y) = x^2 - xy + y^2 + 1$

min = 1

# 14.8 – Constrained Optimization (p. 876)

□ Maximize a function subject to a constraint using Lagrange Multipliers

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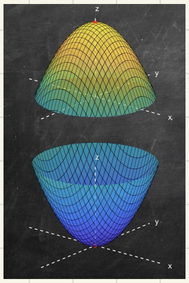
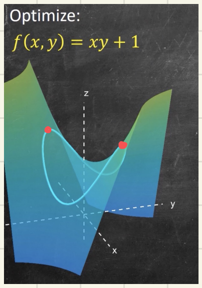
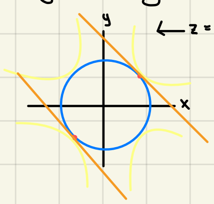
## The Big Idea of Lagrange Multipliers



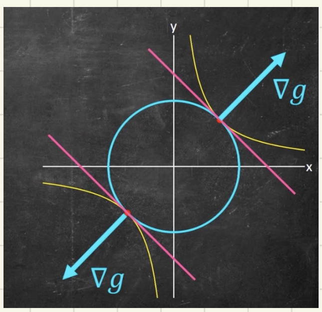




With Constraints:

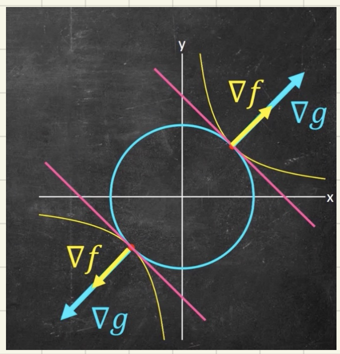
$g(x,y) = x^2 + y^2 - 1 = 0$




ex.




$\longrightarrow$




## Lagrange Multipliers

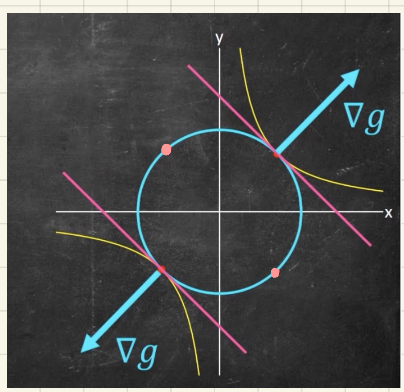
Simultaneously solve...

$$\nabla f = \lambda \nabla g$$
$$g = 0$$

ex. $f(x,y) = xy + 1 \Rightarrow \nabla f = \langle y, x \rangle$

$g(x,y) = x^2 + y^2 - 1 \Rightarrow \nabla g = \langle 2x, 2y \rangle$

$$\boxed{\begin{aligned} y &= \lambda 2x \\ x &= \lambda 2y \\ x^2 + y^2 - 1 &= 0 \end{aligned}}$$




- $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
- $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ max
- $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
- $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ min

$y = \lambda 2(\lambda 2y) = 4\lambda^2 y$

$\Rightarrow y = 0 \Rightarrow x = 0$ not on ellipse!

or $\lambda = \pm \frac{1}{2}$ $\Rightarrow y = \pm x \Rightarrow x^2 + (-x)^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

□ Examples on finding maximum and minimum values of a function, under a constraint, using Lagrange Multipliers

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## Lagrange Multipliers

**Ex.** Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to origin.

Let $(x,y,z)$ be the point on the plane $2x+y-z-5=0$, The distance of this point from the origin is...

$$d(x,y,z) = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$$

Minimize $f(x,y,z) = x^2+y^2+z^2$, such that $2x+y-z-5=0$ $\Rightarrow z = 2x+y-5$

$$\Rightarrow f = x^2+y^2+(2x+y-5)^2$$

$$f_x = 2x + 4(2x+y-5) = 10x+4y-20$$
$$f_y = 2y + 2(2x+y-5) = 4x+4y-10$$

Set $f_x=0$, $f_y=0$ this gives...

$$10x+4y-20 = 0 \quad ①$$
$$4x+4y-10 = 0 \quad ②$$

Equations ① and ② have the solution $x = 5/3$, $y = 5/6$

$$f_{xx} = 10, \quad f_{yy} = 4, \quad f_{xy} = 4$$

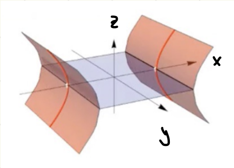
$$D = f_{xx}f_{yy} - f_{xy}^2 = 40-16 > 0$$

$f$ must have a local minimum at $(5/3, 5/6)$.

Using $z = 2x+y-5$ we get $z = -5/6$

The closest point on $2x+y-z-5=0$ from $(0,0)$ is $(5/3, 5/6, -5/6)$

**Ex.** Find the points on the hyperbolic cylinder $x^2-z^2-1=0$ that are closest to the origin.




We want to find the minimum of $f(x,y,z) = x^2+y^2+z^2$
such that $x^2-z^2-1=0$ $\Rightarrow z^2 = x^2-1$

Then... $\qquad g_x = 4, \; g_y = 2y$

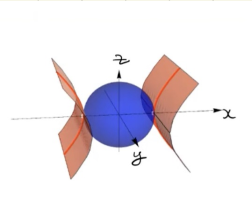
$$g(x,y) = x^2+y^2+(x^2-1)$$
$$= 2x^2+y^2-1$$

Set $g_x = 0$, $g_y = 0$

$\Rightarrow x = 0,\ y = 0$ That's a problem!

---

Let's define ..




$$f(x,y,z) = x^2 + y^2 + z^2 - a^2$$

and

$$g(x,y,z) = x^2 - z^2 - 1$$

We want to find th point of contact where

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 2x, 0, -2z \rangle$$

$$\Rightarrow 2x = \lambda(2x),\ 2y = 0,\ 2z = \lambda(-2z)$$

From th equation

$$2x = \lambda(2x) \Rightarrow x(1-\lambda) = 0$$

either $x=0$ or $\lambda = 1$ ; We know $x \neq 0$ $\Rightarrow$ $\lambda$ must equal 1.

then

$$2z = \lambda(-2z) \Rightarrow 2z = -2z$$

and $z=0$ also, we have $y=0$)

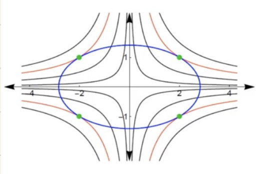
We know the required point is of the form $(x,0,0)$.

Set $g(x,y,z) = 0 \Rightarrow x^2 - z^2 = 1$, with $z=0 \Rightarrow x = \pm 1$.

$(-1,0,0)$ and $(1,0,0)$.

**Ex.** Find th greatest and smallest values that the function $f(x,y) = xy$ takes on th ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$



Suppose $f=1 \Rightarrow y=1/x$
$f=10 \Rightarrow y=10/x$

$f(x,y)=xy$ ; $g(x,y)=x^2/8+y^2/2-1$

We take $\vec{\nabla} f=\lambda\vec{\nabla} g$

$\langle y,x\rangle=\lambda\langle x/4, y\rangle$

$y=\lambda(x/4)$ ; $x=\lambda y$ ; $g(x,y)=0$ , $x^2/8+y^2/2=1$

Using $x=\lambda y$ in $y=\lambda(x/4)$

$\Rightarrow y=\lambda\left(\lambda y/4\right)$
$\Rightarrow y=(\lambda^2/4)y$

Case 1: $y=0 \Rightarrow x=\lambda(0)=0$ ; $(0,0)$ is not on the ellipse!

Case 2: $\lambda=\pm 2 \Rightarrow x=(\pm 2y)$

Now using $x^2/8+y^2/2=1$ ; with $x\pm 2y$

$$\frac{(\pm 2y)^2}{8}+\frac{y^2}{2}=1 \Rightarrow \boxed{y=\pm 1}$$

$y=1 \Rightarrow x=\pm 2$ ;
$y=-1 \Rightarrow x=\pm 2$

Required Points are $(\pm 2,1)$, and $(\pm 2,-1)$

$\boxed{\begin{matrix} f(2,1)=2 \\ f(-2,1)=-2 \end{matrix}}$ $\boxed{y=2/x}$

Ex. The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

$f(x,y,z)=x^2+y^2+z^2$

Subject to : $x+y+z-1=0$ Where $g_1(x,y,z)=x+y+z-1$
$x^2+y^2-1=0$ $g_2(x,y,z)=x^2+y^2-1$

$\vec{\nabla} f=\lambda\vec{\nabla} g_1+\mu\vec{\nabla} g_2$

## 14.8 — Supplemental Materials

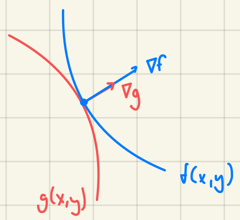
# 14.8 — Lagrange Multipliers (Khan Academy)

## Lagrange Multiplier

Maximize $f(x,y) = x^2y$ on the set $x^2+y^2=1$

$f(x,y) = c$ and $\nabla f$

$g(x,y) = x^2+y^2$ and $\nabla g$




Thus,

$$\nabla f(x_m, y_m) = \lambda \nabla g(x_m, y_m)$$

$\lambda$ ↳ Lagrange Multiplier

We have...

$$\nabla g = \nabla(x^2+y^2) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\nabla f = \nabla(x^2y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$$

$$\begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Finally...

$$\boxed{\begin{array}{r} 2xy = \lambda 2x \\ x^2 = \lambda 2y \\ x^2+y^2-1 = 0 \end{array}}$$

$\longrightarrow x \neq 0, \quad 2y = \lambda 2 \longrightarrow y = \lambda$

$\longrightarrow x^2 = 2y^2 \longrightarrow 2 \cdot 1/3 \longrightarrow \boxed{x = \pm\sqrt{2/3}}$

$\longrightarrow 2y^2 + y^2 - 1 = 0 \longrightarrow 3y^2 = 1 \longrightarrow y^2 = 1/3 \longrightarrow \boxed{y = \pm\sqrt{1/3}}$

Therefore, we have the following points.

$\left(\sqrt{2/3}, \sqrt{1/3}\right) \longrightarrow \boxed{f\left(\pm\sqrt{2/3}, \sqrt{1/3}\right) = (2/3)\left(\sqrt{1/3}\right)}$

$\left(-\sqrt{2/3}, \sqrt{1/3}\right) \nearrow$

$\left(\sqrt{2/3}, -\sqrt{1/3}\right) \leftarrow$ can't be maximum as $-$

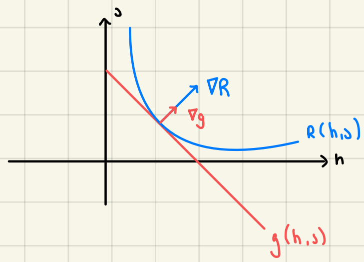
$\left(-\sqrt{2/3}, -\sqrt{1/3}\right) \leftarrow$ can't be maximum as $-$

# Example 1

Labour = \$20/h
Steel = \$2000/ton

Budget : \$20 000

$20h + 2\,000s = 20\,000 \rightarrow g(h,s)$




$R(h,s) = 100 h^{2/3} s^{1/3}$

---

$\nabla R = \lambda \nabla g$

$$\begin{bmatrix} \partial R/\partial h \\ \partial R/\partial s \end{bmatrix} = \begin{bmatrix} 100(2/3) h^{-1/3} s^{1/3} \\ 100(1/3) h^{2/3} s^{-2/3} \end{bmatrix}$$

$$\begin{bmatrix} \partial g/\partial h \\ \partial g/\partial s \end{bmatrix} = \begin{bmatrix} 20 \\ 2000 \end{bmatrix}$$

$$200/3 \left( s^{1/3} / h^{1/3} \right) = \lambda 20 \qquad \mu = s/h$$
$$100/3 \left( h^{2/3} / s^{2/3} \right) = \lambda 2000$$

$$200/3\ \mu^{1/3} = \lambda 20 \longrightarrow \mu^{1/3} = 3/10\ \lambda \quad \mu^{2/3} \longrightarrow \mu = 3/10\ \lambda \mu^{2/3} \times 10 \times 20 \quad \star$$
$$100/3\ \mu^{-2/3} = \lambda 2000 \longrightarrow \mu^{-2/3} = 60\lambda \longrightarrow 1 = 60\lambda \mu^{2/3}$$

$$\star \quad 200\mu = 60\lambda \mu^{2/3}$$
$$1 = 60\lambda \mu^{2/3}$$

$$\longrightarrow 200\mu = 1 \longrightarrow 200\ s/h = 1 \longrightarrow \boxed{h = 200s}$$

$20h + 2\,000s = 20\,000$

$\rightarrow 20(2000s) + 2\,000s = 20\,000$

$\rightarrow 6\,000s = 20\,000$

$\rightarrow \boxed{s = 10/3}$ $\boxed{\begin{aligned} h &= 200(10/3) \\ &= 2\,000/3 \end{aligned}}$

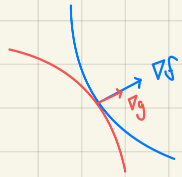
## Lagrangian

$f(x,y) = x^2 e^y \quad y = c$

Max.

$g(x,y) = x^2 + y^2 = b$

$$\boxed{\nabla f = \lambda \nabla g}$$




constant

$$L(x,y,\lambda) = f(x,y) - \lambda\left(g(x,y) - b\right)$$

$$\nabla L = 0$$

Thus

$$\begin{bmatrix} \partial L/\partial x \\ \partial L/\partial y \\ \partial L/\partial \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\partial L/\partial x = \partial f/\partial x - \lambda\, \partial g/\partial x = 0$$
$$\partial L/\partial y = \partial f/\partial y - \lambda\, \partial g/\partial y = 0$$
$$\partial L/\partial \lambda = 0 - (g(x,y) - b) = 0$$
$$\hookrightarrow g(x,y) = b$$

## Meaning of the Lagrange Multiplier

Max

$R(h,s) = \ldots = M^*$

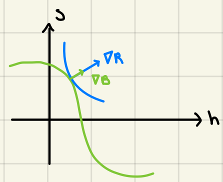
$B(h,s) = \ldots = b$

constant!

$$L(h,s,\lambda) = R(h,s) - \lambda\left(B(h,s) - b\right)$$

$$\nabla L = 0$$

$(h^*, s^*, \lambda^*) =$




$\nabla R = \lambda \nabla B$

$M^* = R(h^*, s^*)$

$M^*(b) = R(h^*(b), s^*(b))$

$\lambda^* = dM^*/db = 2.3$

$M^* \uparrow$ \$2.30

## Proof for the Meaning of Lagrange Multiplier

$\lambda^* = dM^*(b)/db$

$$L(h^*, s^*, \lambda^*) = \underbrace{R(h^*, s^*)}_{M^*} - \lambda^* \underbrace{\cancel{\left(B(h^*, s^*) - b\right)}}_{=0}$$

$$L^*(h^*(b), s^*(b), \lambda^*(b), b) = R(h^*(b), s^*(b)) - \lambda^*\left(B(h^*(b), s^*(b)) - b\right)$$

$$dL^*/db = \cancelto{0}{\partial L/\partial h^*} \cdot dh^*/db + \cancelto{0}{\partial L/\partial s^*} \cdot ds^*/db + \cancelto{0}{\partial L/\partial \lambda^*} \cdot d\lambda^*/db + \partial L/\partial b \cdot \cancelto{1}{db/db}$$

$$= \partial L/\partial b = \lambda^*(b)$$

$$L(h, s, \lambda, b) = R(h, s) - \lambda(B(h, s) - b)$$

Quiz on 14.8

$\boxed{1/3}$ $\boxed{2/3}$

✓ 1. Compute the maximum value (i.e. output) of the function $f(x,y) = xy+1$ constrained to the equation $g(x,y) = x^2+y^2-1=0$.

$f(x,y) = xy+1$ ; $g(x,y) = x^2+y^2-1=0$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$

$y = \lambda 2x$
$x = \lambda 2y$

$g(x,y) = 0$, $x^2+y^2-1=0$

using $x = \lambda 2y$ in $y = \lambda 2x$

$\Rightarrow y = \lambda 2(\lambda 2y)$
$= y = \lambda^2 4y$

$y=0$ $x=0$ $(0,0)$ No!

$\lambda \pm 1/2$ $\Rightarrow x = \pm y$

using $x = \pm y$

$x^2+y^2-1$
$y^2+y^2-1$
$2y^2-1$

$2y^2 = 1$
$y^2 = 1/2$
$y = \pm\sqrt{1/2}$

$y = +\sqrt{1/2}$ $\Rightarrow x = \pm\sqrt{1/2}$

$\boxed{f(\sqrt{1/2}, \sqrt{1/2}) = 1.5}$

X 3. $h(x,y,z) = x-2y+5z$
$f(x,y,z) = x^2+y^2+z^2=30$

$\boxed{\text{minimum}}$

$\langle 1, -2, 5 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$1 = \lambda 2x$
$-2 = \lambda 2y$
$5 = \lambda 2z$

$x = 1/2\lambda$
$y = -\lambda$
$z = 5/2\lambda$

$\lambda = 1/2$
$x = 1$
$y = -1/2$
$z = 5$

$\boxed{h(x,y,z) = 1-2(-1/2)+5(5) = 27}$

✓ $-30$
$(-1, 2, -5)$

X 2. Find the maximum value of the function $g(x,y) = xy+2x$ subject to the constraint $2x+y=30$

$g(x,y) = xy+2x$
$\cancel{h(x,y)} = 2x+y-30$

$y = 30-2x \longrightarrow$

$x(30-2x)+2x$
$30-2x^2+2x$
$-2x^2+2x+30$
$x=$

$\langle y+2, x \rangle = \lambda \langle 2, 1 \rangle$

$y+2 = \lambda 2 \Rightarrow y = \lambda 2 - 2$
$x = 1$ $\lambda = 1$

$\lambda 2 - 2 - y = 0$
$\lambda = 1$, $y = 0$

$\boxed{x=1, y=0 \quad g(x,y)=2}$

$(8, 14)$
$= 128$ ✓