13.1 — Curves in Space & Their Tangents

- Sketch the graph of a vector-valued function
- Compute the tangent line to a curve and explain the idea behind the definition

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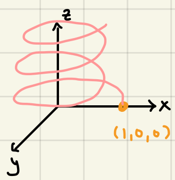
Vector-Valued Function

$\vec{r}(t) = \underbrace{f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}}_{\text{Scalar functions}}$, where $t \in \mathbb{R}$

ex. $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}$

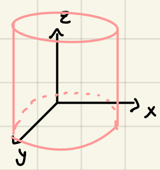
$\vec{r}(0) = \cos(0)\hat{\imath} + \sin(0)\hat{\jmath} + 0\hat{k}$

$= 1\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

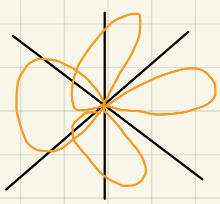


ex. $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}$

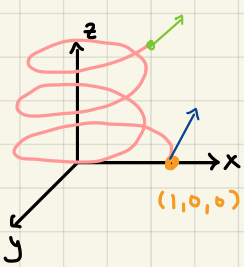
$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$



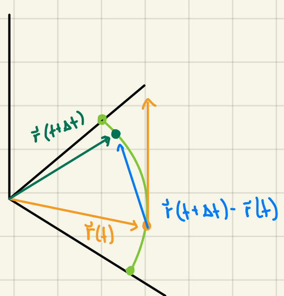
ex. $\vec{r}(t) = \sin(2t)\cos(t)\hat{\imath} + \sin(2t)\sin(t)\hat{\jmath} + \sin(4t)\hat{k}$



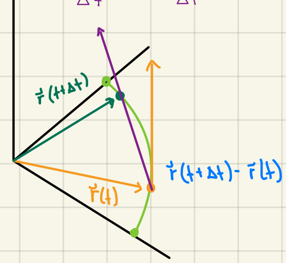
Tangent Vectors



$\longrightarrow$

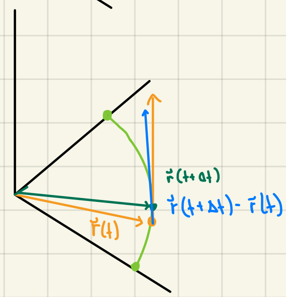


$\frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$



How can we find $\frac{d\vec{r}}{dt}$?

$$\frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$



$$\boxed{\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}}$$

$$\vec{r}(t+\Delta t)-\vec{r}(t) = \left[f(t+\Delta t)\hat{\imath} + g(t+\Delta t)\hat{\jmath} + h(t+\Delta t)\hat{k} \right] - \left[f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k} \right]$$

$$= \left[f(t+\Delta t) - f(t)\right]\hat{\imath} + \left[g(t+\Delta t) - g(t)\right]\hat{\jmath} + \left[h(t+\Delta t) - h(t)\right]\hat{k}$$

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]\hat{\imath} + \lim_{\Delta t \to 0} \left[\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]\hat{\jmath}$$

$$+ \lim_{\Delta t \to 0} \left[\frac{h(t+\Delta t) - h(t)}{\Delta t}\right]\hat{k}$$

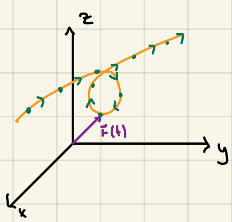
$$\boxed{\frac{d\vec{r}}{dt} = \frac{df}{dt}\hat{\imath} + \frac{dg}{dt}\hat{\jmath} + \frac{dh}{dt}\hat{k}} \quad \text{or} \quad \boxed{\vec{r}\,'(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}}$$

Leibniz's Notation — Lagrange's Notation

- Curves in Space as Vector-Valued functions
- Limit, Continuity and Differentiability of Such Curves
- Velocity, Speed, Acceleration and Direction of Motion

A curve in space can be represented in Vector form as...

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \; t \in \mathbb{R}.$$



$y = f(x)$
$f(x,y) = 0$

- It describes the motion of a particle in space.

The functions $f, g,$ and h are called component functions. $\vec{r}(t)$ is a vector-valued function.
The domain of a vector-valued function is the common domain of all its component functions.

Recall: $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$

$$= \langle f(t), g(t), h(t) \rangle$$

Example: $\vec{r}(t) = \langle 1, 1, t \rangle$ (z is changing linearly as a function of t)
(xy-plane is fixed)

Example: $\vec{r}(t) = \langle t, t, t \rangle$

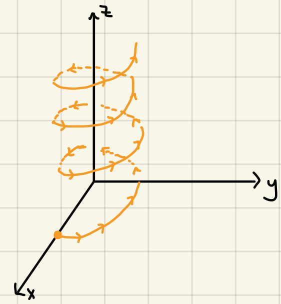
Example 1: Graph the vector function $r(t) = (\cos(t))i + (\sin(t))j + tk$

We can re-write this vector-valued function as...

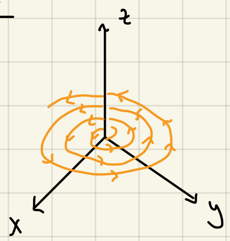
$$\vec{r}(t) = \langle \underset{x(t)}{\cos(t)}, \underset{y(t)}{\sin(t)}, \underset{z(t)}{t} \rangle$$

$$\boxed{x^2 + y^2 = 1}$$

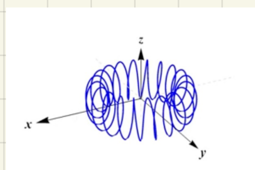
$$\cos^2(t) + \sin^2(t) = 1$$



Example



$$\vec{r}(t) = \langle t\cos(t), t\sin(t), t \rangle$$



$$\vec{r}(t) = \langle (4 + \sin(20t))\cos(t), (4 + \sin(20t))\sin(t), \cos(20t) \rangle$$

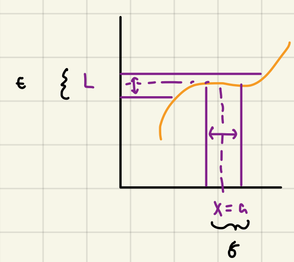
Limits and Continuity

Let $\vec{r}(t)$ be a vector function w/ domain D, and let $\vec{L}$ be another vector. We say that $\vec{r}$ has a limit $\vec{L}$ as $t \to t_0$ and write

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if for every $\epsilon > 0$, there exists a corresponding $\alpha > 0$ such that for all $t \in D$.

$$|\vec{r}(t) - \vec{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \alpha$$



$\lim_{x \to a} f(x) = L$

If $\vec{L} = \langle L_1, L_2, L_3 \rangle$, then if $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$, then...

$$\lim_{t \to t_0} f(t) = L_1, \quad \lim_{t \to t_0} g(t) = L_2 \quad \lim_{t \to t_0} h(t) = L_3$$

A vector function $\vec{r}(t)$ is continuous at a point $t = t_0$ in its domain if...

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

The function $\vec{r}(t)$ is continuous if it is continuous at every point in its domain.

$$\lim_{x \to a} f(x) = f(a)$$

Example 2: For the vector function $r(t) = \cos(t)i + \sin(t)j + tk$, determine the limit as t approaches $\pi/4$.

$$\lim_{t\to\pi/4} \vec{r}(t) = \lim_{t\to\pi/4} \langle \cos(t), \sin(t), t \rangle = \left\langle \lim_{t\to\pi/4} \cos(t), \lim_{t\to\pi/4} \sin(t), \lim_{t\to\pi/4} t \right\rangle$$

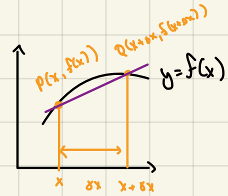
$$= \langle \sqrt{2}/2, \sqrt{2}/2, \pi/4 \rangle$$

Example 3: Discuss the continuity of the vector function $r(t) = \cos(t)i + \sin(t)j + \lfloor t \rfloor k$. Where $\lfloor t \rfloor$ is the greatest integer function.

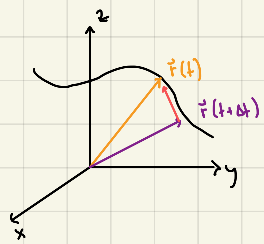
This vector valued function is discontinuous at every integer value because of $\lfloor t \rfloor$.

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$; continuous curve. $0 \le t \le 2\pi$.

Differentiation



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x-\delta x) - x}$$



$$\lim_{\Delta t\to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

or $$\frac{d\vec{r}}{dt} = \langle df/dt, dg/dt, dh/dt \rangle$$

Supposing $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

The vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ has a derivative at t if $f(t), g(t)$, and $h(t)$ have a derivative at t. And, the derivative is the function:

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

A vector $\vec{r}(t)$ is differentiable if it is differentiable at every point of its domain.

$\vec{r}(t)$ is smooth if $d\vec{r}/dt$ is continuous and never $\vec{0}$, that is, $f(t), g(t)$, and $h(t)$ have continuous first derivatives and are not simultaneously zero.

A curve made up of a finite number of smooth curves pieced together in a continuous fashion is called a piecewise smooth curve

For the Motion of a Particle:

If $\vec{r}(t)$ defines the motion of a particle.

i) Velocity: $\vec{v}(t) = d\vec{r}/dt$

ii) Speed: $s = |\vec{v}(t)|$

iii) Acceleration: $\vec{a}(t) = d\vec{v}/dt$

iv) Direction of Motion: $\vec{v}/|\vec{v}|$

Example 4 Find velocity, speed, acceleration of $r(t) = (2\cos(t))i + (2\sin(t))j + (5\cos^2(t))k$

Velocity: $\vec{v}(t) = d\vec{r}/dt = \langle -2\sin(t), 2\cos(t), -10\cos(t)\sin(t) \rangle$

$= \langle -2\sin(t), 2\cos(t), -5\sin(2t) \rangle$

$\vec{a}(t) = d\vec{v}/dt = \langle -2\cos(t), -2\sin(t), -10\cos(2t) \rangle$

Speed: $s = |\vec{v}(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t) + 25\sin^2(2t)}$

$= \sqrt{4 + 25\sin^2(2t)}$

Direction of Motion: $\vec{v}/|\vec{v}| = \dfrac{2}{\sqrt{4 + 25\sin^2(2t)}} \langle -\sin(t), \cos(t), -5/2\sin(2t) \rangle$

- Rules of Differentiation
- Vector Functions of Constant Length

Rules of Differentiation, for $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $t \in \mathbb{R}$

- $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$
- $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$
- $d\vec{r}/dt = \langle df/dt, dg/dt, dh/dt \rangle$
- $\vec{v}(t) = d\vec{r}/dt$, $\vec{a}(t) = d^2\vec{r}/dt^2 = d\vec{v}/dt$, speed $= |\vec{v}(t)|$

Let $\vec{u}, \vec{v}$ be differentiable vector functions of t, $\vec{c}$ a constant vector and f is a differentiable function of t.

- $\frac{d}{dt}[\vec{c}] = \vec{0}$; constant function Rule.
- $\frac{d}{dt}[\alpha \vec{u}(t)] = \alpha\, d\vec{u}/dt$; α is scalar ; Scalar Multiplication

- $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$.

- $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u}\,'(t) \pm \vec{v}\,'(t)$

- $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \vec{u}\,'(f(t)) \cdot f'(t)$; Chain Rule

- $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$; Dot Product Rule

- $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$; Order Matters! ; Cross Product Rule

Dot Product Rule:

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\frac{d}{dt}\left[\vec{u} \cdot \vec{v}\right] = \frac{d}{dt}\left[u_1v_1 + u_2v_2 + u_3v_3\right]$$

$$= u_1'v_1 + u_1v_1' + u_2'v_2 + u_2v_2' + u_3'v_3 + u_3v_3'$$

$$= \left(u_1'v_1 + u_2'v_2 + u_3'v_3\right) + \left(u_1v_1' + u_2v_2' + u_3v_3'\right)$$

$$= \vec{u}\,' \cdot \vec{v} + \vec{u} \cdot v'$$

Cross-Product Rule:

$$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) \times \vec{v}(t+\Delta t) - \vec{u}(t) \times \vec{v}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) \times \vec{v}(t+\Delta t) - \vec{u}(t) \times \vec{v}(t+\Delta t) + \vec{u}(t) \times \vec{v}(t+\Delta t) - \vec{u}(t) \times \vec{v}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\vec{u}(t+\Delta t) - \vec{u}(t)\right] \times \vec{v}(t+\Delta t) + \vec{u}(t) \times \left[\vec{v}(t+\Delta t) - \vec{v}(t)\right]}{\Delta t}$$

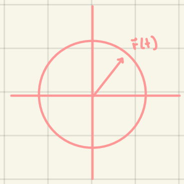
$$= \lim_{\Delta t \to 0} \left[\frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}\right] \times \vec{v}(t+\Delta t) + \vec{u}(t)\left[\frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}\right]$$

$$= \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$$

Vector Functions of Constant Length

Suppose we have $\vec{r}(t)$ such that $|\vec{r}(t)| = c$.

ex.



$|\vec{r}(t)| = c$
$= r$

Now Consider

$\vec{u}\cdot\vec{u} = |\vec{u}|^2$

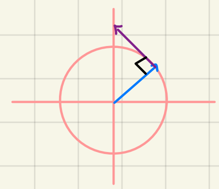
$$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$$

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

$|\vec{r}(t)| = 1.$

Differentiating both sides...

$$\Rightarrow \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \frac{d}{dt}c^2$$



$$\Rightarrow \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 0$$
$$\Rightarrow 2\vec{r}\,'(t)\cdot\vec{r}(t) = 0$$
$$\Rightarrow \vec{r}\,'(t)\cdot\vec{r}(t) = 0$$

Therefore the vectors $\vec{r}(t)$ and $\vec{r}\,'(t)$ are both orthogonal to each other.

Note: If $\vec{r}(t)$ is a differentiable vector function of t, and the length of $\vec{r}(t)$ is constant, then ...

$$\boxed{\vec{r}\,'(t)\cdot\vec{r}(t) = 0} \quad \text{or} \quad \boxed{\vec{r}(t)\cdot\frac{d\vec{r}}{dt} = 0}$$

13.2 – Integrals of Vector Functions

- Integrals of Vector Functions
- We can find the Position function for a moving object using its Acceleration
- Position Function for an Ideal Projectile Motion

The Indefinite Integral

The indefinite integral of a vector valued function $\vec{r}(t)$ with respect to t is the set of all anti-derivatives of $\vec{r}(t)$, denoted by $\int \vec{r}(t)\,dt$. If $\vec{R}(t)$ is an antiderivative of $\vec{r}(t)$, then...

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{c}$$

Example:

$$\int \langle \cos t, 1, -2t \rangle\,dt$$

$$= \langle \sin t, t, -t^2 \rangle + \vec{c}$$

$$\boxed{\int \langle f(t), g(t), h(t) \rangle\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle + \vec{c}}$$

The Definite Integral

If the components of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ are integrable on some domain for $t([a,b])$, then so is $\vec{r}(t)$, and...

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

Example:

$$\int_0^{\pi} \langle \cos(t), 1, -2t \rangle\,dt$$

$$= \left\langle \sin(t)\Big|_0^{\pi}, t\Big|_0^{\pi}, -t^2\Big|_0^{\pi} \right\rangle$$
$$= \langle 0, \pi, -\pi^2 \rangle$$

By Fundamental Theorem of Calculus (for continuous vector functions).

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b$$
$$\vec{R}(b) - \vec{R}(a)$$

$$\int_a^b f(x)\,dx$$
$$= F(b) - F(a)$$

Where $\vec{R}(t)$ is an anti-derivative of $\vec{r}(t)$. i.e., $\vec{R}'(t) = \vec{r}(t)$.

Example 3

$a(t) = -(3\cos(t))i - (3\sin(t))j + 2k.$

$t=0$, $P(4,0,0)$, $v(0) = 3j$

Position as a function of time t.

We have $\vec{a}(t) = \langle -3\cos t, -3\sin t, 2 \rangle$

$\Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \langle -3\sin t, 3\cos t, 2t \rangle + \vec{c}$

at $t=0$ $\quad \vec{v}(0) = \vec{V}_0 = \langle 0,3,0 \rangle = \langle 0,3,0 \rangle + \vec{c}_1 \quad \Rightarrow \vec{c} = \vec{0}$

$\Rightarrow \vec{v}(t) = \langle -3\sin t, 3\cos t, 2t \rangle$

$\Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt = \langle 3\cos t, 3\sin t, t^2 \rangle + \vec{c}_2$

We know that $\vec{r}(0) = \langle 4,0,0 \rangle$

$\Rightarrow \langle 4,0,0 \rangle = \vec{r}(0) = \langle 3,0,0 \rangle + \vec{c}_2$

$\Rightarrow \langle 4,0,0 \rangle = \langle 3,0,0 \rangle + \vec{c}^2$

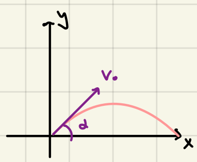
$\Rightarrow \vec{c}_2 = \langle 1,0,0 \rangle$

$\Rightarrow \vec{r}(t) = \langle 3\cos t, 3\sin t, t^2 \rangle + \langle 1,0,0 \rangle$

or $\vec{r}(t) = \langle 3\cos t + 1, 3\sin t, t^2 \rangle$

Vector & Parametric Equations for Ideal Projectile Motion

We assume a projectile motion starting at $t=0$ from 0, with initial velocity $\vec{V}_0$; making an angle α with the horizontal.



As we start from 0, $\vec{r}_0 = \langle 0,0 \rangle$

$\vec{V}_0 = \langle |\vec{V}_0|\cos\alpha, |\vec{V}_0|\sin\alpha \rangle$

We have $\vec{F} = m\vec{a} \quad \Rightarrow \quad m\dfrac{d^2\vec{r}}{dt^2} = -m(g)\hat{j}$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \langle 0, -g \rangle$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \langle 0, -gt \rangle + \vec{V}_0 = \langle 0, -gt \rangle + \langle |\vec{V}_0|\cos\alpha, |\vec{V}_0|\sin\alpha \rangle$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \langle |\vec{V}_0|\cos\alpha, |\vec{V}_0|\sin\alpha - gt \rangle$$

Integrate again!

$$\Rightarrow \vec{r}(t) = \langle (|\vec{v}_0| \cos\alpha)t, (|V_0| \sin\alpha)t - gt^2/2 \rangle + \vec{r}_0$$

$$\Rightarrow \vec{r}(t) = \langle (|\vec{v}_0| \cos\alpha)t, (|\vec{v}_0| \sin\alpha)t - gt^2/2 \rangle$$

In Components:

$$x(t) = (|\vec{v}_0| \cos\alpha)t$$
$$y(t) = (|\vec{v}_0| \sin\alpha)t - gt^2/2$$

Example 4

Speed 500 m/sec, angle 60° from horizontal
Find $t = 10$. where $g = 9.8$

Here $|\vec{v}_0| = 500$ m/sec, $\alpha = 60°$, $\vec{r}(t)\Big|_{t=10s} = ?$

$$\vec{r}(t) = \langle (|\vec{v}_0| \cos\alpha)t, (|\vec{v}_0| \sin\alpha)t - gt^2/2 \rangle$$

$$\vec{r}(10) = \langle 2500, 3840 \rangle$$

100%

Quiz on 13.1 / 13.2

✓ 1. helix $r(t) = \cos(t)i + \sin(t)j + 2tK$

$dr/dt = -\sin(t), \cos(t), 2$

$t = \pi/2$

$\langle -1, 0, 2 \rangle$

✓ 2. $r(t) = t^2 i + e^t j - 2\cos\pi t\, k$

$\lim_{t \to 0} = \langle 0, 1, -2 \rangle$

✓ 3. $r'(t) = 3i + 2tj$ find $r(5)$

$r(1) = 2i + 5j$

$r(t) = 3t + t^2 + C$

$t=1 \quad \langle 3, 1 \rangle + \vec{c}_1$

$t=1 \quad \langle 2, 5 \rangle$

$\langle 2, 5 \rangle = \langle 3, 1 \rangle + \vec{c}_1$

$\langle -1, 4 \rangle = \vec{c}_1$

$r(t) = (3t, t^2) + \langle -1, 4 \rangle$

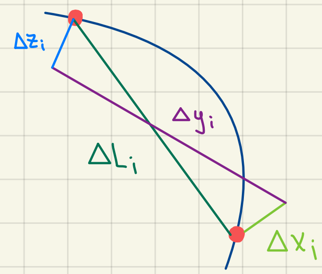
$r(5) = \langle 3(5) - 1, 5^2 + 4 \rangle$

$= \langle 14, 29 \rangle$

13.3 — Arc Length of Curves

- Illustrate the idea behind the arc length formula
- Compute the arc length of a curve

Deriving the Arc Length Formula



What is the arclength of the curve $\vec{r}(t)$ for $a \le t \le b$?

1. Break $[a,b]$ into n segments (n equal segments)

$$\text{Arclength} \approx \sum_{i=1}^{n} (\text{Length of } i^{th} \text{ segment})$$

2. Compute length of i^{th} segment

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$$\text{Arclength} \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \; \frac{\Delta t}{\Delta t}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \; \Delta t$$

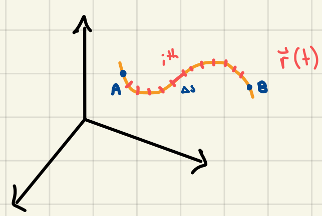
$$\text{Arclength} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \; \Delta t$$

for $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ on $[a,b]$

$$\boxed{\text{Arclength} = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\; dt}$$

- Arc Length Formula
- Arc Length Parametrization

Arc Length in Space



Suppose we divide the curve into n-sub segments.

$$\Delta s \approx \vec{r}(t_{i+1}) - \vec{r}(t_i), \quad t_{i+1} = t_i + \Delta t$$

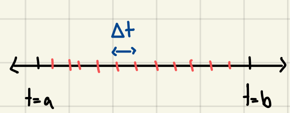
$$= \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \cdot \Delta t$$

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t}\right] \Delta t$$

$$= \int_a^b d\vec{r}/dt \; \Delta t$$

Length of the curve from A to B,

$$\vec{r}(t)\Big|_{t=a} = A, \quad \vec{r}(t)\Big|_{t=b} = B$$



$$s \approx \sum_{i=1}^{n} \left[\frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t}\right] \Delta t$$

The length of a smooth curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ that is traced exactly once as t increases from $t=a$ to $t=b$, is given by...

$$\boxed{S = \int_a^b \left| d\vec{r}/dt \right| dt \quad \underline{\text{or}} \quad \int_a^b |\vec{v}(t)|\, dt \quad \underline{\text{or}} \quad \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt}$$

<u>Example 1</u>: A glider is soaring upward along the helix $r(t) = (\cos t)i + (\sin t)j + tk$. How long is the glider's path from $t=0$ to $t=2\pi$.

$$S = \int_0^{2\pi} \left| d\vec{r}/dt \right| dt = \int_0^{2\pi} \left| \langle -\sin t, \cos t, 1 \rangle \right| dt$$
$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\; dt$$
$$= \int_0^{2\pi} \sqrt{2}\, dt$$
$$= \sqrt{2}\,(2\pi)$$

Arclength $= s = \sqrt{2}\,(2\pi)$ unit of length.

<u>Exercise</u>: Find the length of the curve $r(t) = i + t^2 j + t^3 k$ for $0 \le t \le 1$.
$\vec{r}(t) = \langle 1, t^2, t^3 \rangle$

$$S = \int_0^1 \left| d\vec{r}/dt \right| dt = \int_0^1 \langle 0, 2t, 3t^2 \rangle\, dt$$
$$= \int_0^1 \sqrt{0 + 4t^2 + 9t^4}\; dt$$
$$= \int_0^1 2t\sqrt{1 + 9/4\, t^2}\; dt$$

let $u = 3t/2$, $du = 3/2\, dt$

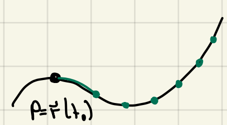
$$\Rightarrow S = 2/3 \int_0^{3/2} (4u/3)\sqrt{1+u^2}\; du$$

$$= 2^2/3^2 \int_0^{3/2} (1+u^2)^{1/2}\,(2u)\, du$$
$$= 2^2/3^2 \left[\frac{(1+u^2)^{3/2}}{3/2} \right]\Bigg|_0^{3/2} = 4/9\left[(1 + 9/4)^{3/2} / 3/2 - 1/3/2 \right]$$
$$\approx 1.44$$

<u>Now Consider</u>:

$$S(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$$

$t_0 \rightarrow$ constant



S is the arclength parameter, with base point $\vec{r}(t_0)$

$\underline{\text{Exercise}}$: Find the arc-length parametrization for the curve $r(t) = 3\sin t\, i + 4t\, j + 3\cos t\, k$ for $0 \le t \le 1$.

$$\vec{r}(t) = \langle 3\sin t, 4t, 3\cos t \rangle$$

$$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau, \quad \vec{v}(t) = d\vec{r}/dt = \langle 3\cos t, 4, -3\sin t \rangle$$

$$|\vec{v}(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t}$$
$$= \sqrt{9+16}$$
$$= 5$$

$$\Rightarrow s(t) = \int_0^t 5\, d\tau$$
$$= [5\tau]_0^t$$

$$\Rightarrow \boxed{s(t) = 5t} \Longrightarrow t = s/5$$

Now the length of the curve from $t=0$ to $t=1$ is...

$$\boxed{s(1) = 5}$$

From $t=0$ to $t=2$ is...

$$\boxed{s(2) = 5(2) = 10}$$

our curve is defined as... $\vec{r}(t) = \langle 3\sin t, 4t, 3\cos t \rangle$

re-parametrized curve is given by..

$$\vec{r}(s) = \langle 3\sin(s/5), 4s/5, 3\cos(s/5) \rangle$$

$$s(t) = \int_{t_0}^t |\vec{v}(\tau)\, d\tau$$

$\underline{\text{Example}}$: Find the arc-length parametrization for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ with $t_0 = 0$.

$$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t |\langle -\sin\tau, \cos\tau, 1 \rangle|\, d\tau$$
$$= \int_0^t \sqrt{\sin^2\tau + \cos^2\tau + 1}\, d\tau$$
$$= \int_0^t \sqrt{2}\, d\tau$$
$$= \sqrt{2}\, t$$

therefore... $t = s/\sqrt{2}$

$$\vec{r}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2} \rangle$$

Speed on a Smooth Curve

We had $s(t) = \int_0^t \underbrace{|\vec{v}(\tau)|}_{\text{speed}} d\tau$, taking derivative w/ respect to t

$$\Rightarrow ds/dt = |\vec{v}(t)| > 0$$

* So, s is always an increasing function of t.

Unit Tangent

$$d\vec{r}/ds = d\vec{r}/dt \cdot dt/ds$$

$$= \frac{d\vec{r}/dt}{ds/dt}$$

$$= \vec{v}(t) / \underline{|\vec{v}(t)|} > 0.$$

$$= \vec{T}, \text{ unit tangent}$$

Once we have represent $\vec{r}$ as a function of s.

$$\vec{r} = \vec{r}(s)$$

Then th derivative w/ respect to s is always a unit tangent.

$d\vec{r}/ds$ is a unit tangent.

Example: $\vec{r}(t) = \langle 1+3\cos t, 3\sin t, t^2 \rangle$

- $\vec{v}(t) = d\vec{r}/dt = \langle -3\sin t, 3\cos t, 2t \rangle$

$$|\vec{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$
$$= \sqrt{9+4t^2}$$

$$\text{unit tangent} = \vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{9+4t^2}} \langle -3\sin t, 3\cos t, 2t \rangle$$

$$s = \int_0^t |\vec{v}(\tau)| d\tau, \ t = f(s) \Rightarrow d\vec{r}/ds$$

Summary

- $S = \int_a^b |d\vec{r}/dt| \, dt$
- $s(t) = \int_a^b |\vec{v}(\tau)| \, d\tau$
- $ds/dt = |\vec{v}(t)| > 0$
- $d\vec{r}/ds =$ is a unit tangent

Quiz on 13.3

$\boxed{100\%}$

✓ 1. Arc length of $r(t) = \cos(t)i + \sin(t)j + \sqrt{2}t\,K$ on $[0, \pi]$.

$$s = \int_0^\pi |d\vec{r}/dt|\,dt = \int_0^\pi |\langle -\sin t, \cos t, \sqrt{2} \rangle|\,dt$$

$$= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 2}\;dt$$

$$= \int_0^\pi \sqrt{3}\;dt$$

$$= \sqrt{3}\pi \approx 5.44139.. \longrightarrow 5.4$$

✓ 2. Space Curve $r(t) = t_i + \frac{1}{2}t^2 j + \frac{1}{3}t^3 K$

$T(t)$ when $t = 0$.

$\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$

$\langle 1, t, t^2 \rangle$

$\sqrt{1 + t^2 + t^4} \quad \langle 1/\sqrt{}, t/\sqrt{}, t^2/\sqrt{} \rangle$

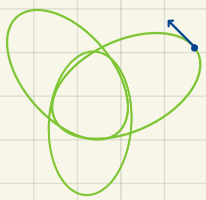
$= \langle 1, 0, 0 \rangle$

13.4 — Curvature

- Compute the curvature at a point given a choice of parameterization
- Compute Principle Unit Normal at a point
- Sketch the Circle of Curvature

Curvature, Normal Vectors, and Circle of Curvature

How do we measure "curvature" at a point?



We can look at how the tangent vector changes!

Definition: For $\vec{T}$ the unit tangent to a smooth curve, the curvature function κ is given by...

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

When parameterized by some t not arclength s:

$$\kappa = \left| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right|$$

For $|\vec{v}| = ds/dt$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Example: Circle of radius a: $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}$

$\vec{v} = d\vec{r}/dt = -(a\sin t)\hat{i} + (a\cos t)\hat{j} \quad \Rightarrow |\vec{v}| = a$

$\vec{T} = \vec{v}/|\vec{v}| = -(\sin t)\hat{i} + (\cos t)\hat{j}$

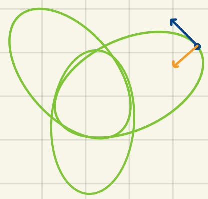
$d\vec{T}/dt = -(\cos t)\hat{i} - (\sin t)\hat{j}$

$\left| d\vec{T}/dt \right| = 1$

$$\boxed{\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| = \frac{1}{a}}$$

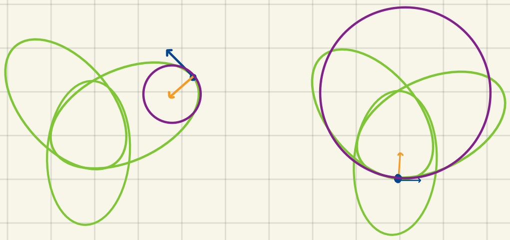
Definition: When $\kappa \neq 0$, the principle unit normal vector for a smooth curve in the plane is...

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$



Definition: Circle of Curvature

1. Tangent to the curve at the point
2. Same curvature as the curve (i.e. radius = $1/\kappa$)
3. Center on the concave (or inner) side



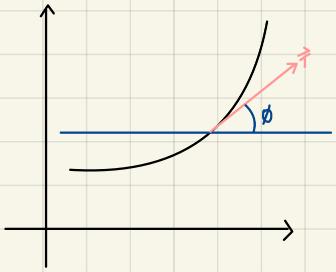
- ○ Curvature
- ○ Vector Principle Unit Normal

Curvature and Normal Vectors of a Curve

If $\vec{T} = d\vec{r}/ds$ is the unit vector of a smooth curve, the curvature function of the curve is...

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

* The rate at which $\vec{T}$ turns per unit of arc-length along the curve.



Suppose $\vec{T} = \langle \cos\phi, \sin\phi \rangle$

$\Rightarrow d\vec{T}/ds = \langle -\sin\phi, \cos\phi \rangle \, d\phi/ds$

$\Rightarrow |d\vec{T}/ds| = \left|\frac{d\phi}{ds}\right|$

- If $\left|\frac{d\vec{T}}{ds}\right|$ is large, $\vec{T}$ turns sharply, the curvature is large.
- If $\left|\frac{d\vec{T}}{ds}\right|$ is closer to zero, curvature is small

$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right| = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}$$

; $\vec{T} = \vec{v}/|\vec{v}|$ is a unit vector and $\vec{v}(t) = d\vec{r}/dt$.

* $\vec{r}(t)$ is a smooth curve (magnitude ($|\vec{v}|$) is never zero)!

Example. Suppose $\vec{r}(t) = \vec{r}_0 + t\vec{d}$; a straight line.

$\vec{d}$ — direction vector!

$$\vec{v}(t) = d\vec{r}/dt = \vec{d}$$

$$T = \vec{v}/|\vec{v}| \ , \ \kappa = 1/|\vec{v}| \left| d\vec{T}/dt \right| , \ d\vec{T}/dt = \vec{0}$$; No bending!

$$\boxed{\Rightarrow \kappa = 0}$$

Example: Now consider $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$, circle of radius a.

$$\vec{v} = d\vec{r}/dt = \langle -a\sin t, a\cos t \rangle, \ |\vec{v}| = a$$

$$\Rightarrow \vec{T} = \vec{v}/|\vec{v}| = \langle -\sin t, \cos t \rangle$$

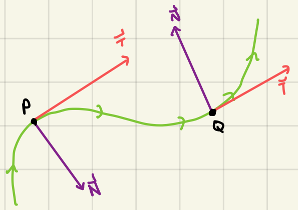
$$\Rightarrow d\vec{T}/dt = \langle -\cos t, -\sin t \rangle, \ |d\vec{T}/dt| = 1$$

$$\Rightarrow K = \frac{|d\vec{T}/dt|}{|\vec{v}|} = 1/a\,(1) = 1/a$$

$$\boxed{\Rightarrow K = 1/a}$$; bending depends on the radius a.

At a point where $K \neq 0$, The Principal Unit Normal vector for a smooth curve is...

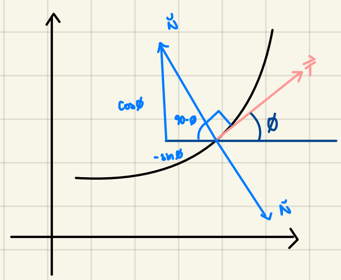
$$\vec{N} = 1/K \ d\vec{T}/ds = \frac{1}{|d\vec{T}/ds|} \cdot d\vec{T}/ds$$



$$\vec{v} = \langle x, y \rangle$$

$$\vec{v}_\perp = \langle -y, x \rangle$$

$$\vec{v}_\perp = \langle y, -x \rangle$$



$$\vec{T} = \langle \cos\phi, \sin\phi \rangle$$

$$d\vec{T}/ds = \langle -\sin\phi, \cos\phi \rangle \, d\phi/ds$$

* $\vec{N} = \frac{1}{|d\vec{T}/ds|} \cdot d\vec{T}/ds$ points towards the concave side of the curve.

$$\vec{N} = \frac{1}{|d\vec{T}/ds|} \cdot \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt \cdot dt/ds}{|d\vec{T}/dt \cdot dt/ds|} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\boxed{\vec{N} = \frac{1}{\left|\frac{d\vec{T}}{ds}\right|} \cdot \frac{d\vec{T}}{ds} \ , \quad \vec{N} = \frac{1}{|d\vec{T}/dt|} \cdot \frac{d\vec{T}}{dt}}$$

Since $\vec{T}$ is a unit vector, $\vec{T}\cdot\vec{T} = 1$

$\Rightarrow d\vec{T}/ds \cdot \vec{T} + \vec{T} \cdot d\vec{T}/ds = 0$

$\Rightarrow \vec{T} \cdot \underbrace{d\vec{T}/ds}_{\text{direction of } \vec{N}} = 0$

<u>Example</u>: $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$, $a, b \geq 0$, $a^2+b^2 \neq 0$.

$\vec{v}(t) = d\vec{r}/dt = \langle -a\sin t, a\cos t, b \rangle$

$\Rightarrow \vec{T} = \vec{v}/|\vec{v}| = 1/\sqrt{a^2+b^2}\,\langle -a\sin t, a\cos t, b \rangle$ $\Rightarrow d\vec{T}/dt = 1/\sqrt{a^2+b^2}\,\langle -a\cos t, -a\sin t, 0 \rangle$

$\Rightarrow |d\vec{T}/dt| = a/\sqrt{a^2+b^2}$

$\Rightarrow \boxed{\kappa = |d\vec{T}/dt|/|\vec{v}| = a/a^2+b^2}$

curvature decreases as we increase b for a fixed a.

$d\vec{T}/dt = 1/\sqrt{a^2+b^2}\,\langle -a\cos t, -a\sin t, 0 \rangle$, $|d\vec{T}/dt| = a/\sqrt{a^2+b^2}$

$$\boxed{\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \langle -\cos t, -\sin t, 0 \rangle}$$

<u>Summarize</u>

- $\kappa = |d\vec{T}/ds|$, $\kappa = 1/|\vec{v}|\;|d\vec{T}/dt|$
- $\vec{v}(t) = d\vec{r}/dt$, $\vec{T} = \vec{v}/|\vec{v}|$
- $\vec{N} = 1/\kappa \cdot d\vec{T}/ds = d\vec{T}/ds \,/\, |d\vec{T}/ds|$
- $\vec{N} = d\vec{T}/dt \,/\, |d\vec{T}/dt|$; it always points towards the concave side.
- $\boxed{\vec{T}\cdot\vec{N} = 0}$

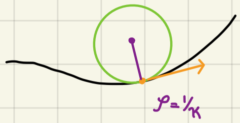
- Osculating Circle (AKA the Circle of Curvature, from above)
- Center and Radius of the Osculating Circle

The <u>circle of Curvature</u> (Osculating Circle) at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that...

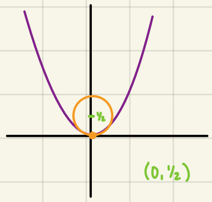
1. It is tangent to the curve at P. (Has the same tangent at P as the curve).
2. Has the same curvature as curve at P.
3. The center lies towards the concave of the inner side of the curve.

The <u>Radius of Curvature</u> of the curve at P is the radius of the circle of curvature, which is...

Radius of Curvature $= \rho = 1/\kappa$



<u>Example</u>: Find the osculating circle of the parabola $y = x^2$ at $(0,0)$.



$\vec{r}(t) = \langle t, t^2 \rangle$

$\vec{v} = d\vec{r}/dt = \langle 1, 2t \rangle$

$|\vec{v}| = \sqrt{1+4t^2}$

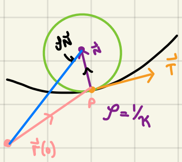
$\vec{T} = \vec{v}/|\vec{v}| = \left\langle \dfrac{1}{\sqrt{1+4t^2}}, \dfrac{2t}{\sqrt{1+4t^2}} \right\rangle$

$\kappa\Big|_{t=0} = 1/|\vec{v}(0)| \left| d\vec{T}/dt \Big|_{t=0} \right| \stackrel{\dots}{=} 2$

This implies that the radius of the osculating circle $= \rho = 1/\kappa = 1/2$

<u>Required Equation is</u> $(x-0)^2 + (y-1/2)^2 = (1/2)^2$

<u>Example</u>:

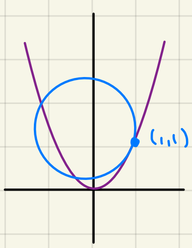


From the figure...

Center: $\vec{r}(0) + \rho(0)\vec{N}$

Radius: $\rho = 1/\kappa$

<u>Example</u>: Determine the circle of curvature to the parabola $y = x^2$ at $(1,1)$



$\vec{r}(t) = \langle t, t^2 \rangle$

$\kappa\Big|_{t=1} = 2/5\sqrt{5} \Rightarrow \rho = 5\sqrt{5}/2$

Centre at $(1,1) = \vec{r}(1) + \rho(1)\vec{N}(1)$

$= (-4, 7/2)$

$$\boxed{(x+4)^2 + (y - 7/2)^2 = 125/4}$$

Quiz on 13.4

100%

✓ 1. $r(t) = \sin(t)i + \cos(t)j + 3k$
curvature @ $t=2$.

$$K = \left| \frac{d\vec{T}}{ds} \right|$$

$\langle \sin t, \cos t, 3 \rangle$

$\rightarrow \langle \cos t, -\sin t, 0 \rangle$

$\sqrt{\cos^2 t + \sin^2 t + 0^2} = \sqrt{1} = 1$

$\rightarrow \langle -\sin t, -\cos t, 0 \rangle$

$\sqrt{\sin^2 t + \cos^2 t + 0^2} = \sqrt{1} = 1$

$\rightarrow K = 1/1$

✓ 2. $r(t) = ti + \frac{1}{2}t^2 j + \frac{1}{3}t^3 K$

$\vec{N}$ @ $t=0$.

$\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$

$\langle 1, t, t^2 \rangle$

$\sqrt{1^2 + t^2 + t^4} = \sqrt{1} = 1$

$\langle 0, 1, t \rangle$

$\langle 0, 1, 0 \rangle$

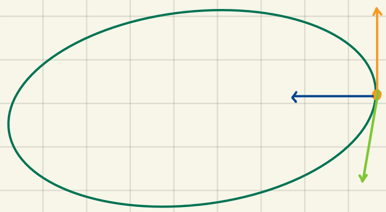
13.5 – Tangential and Normal Components of Acceleration

- ☐ Tangent, Normal, and Binorval Vector (TNB or Frenet Frame)
- ☐ Compute th TNB frame at point

Three Vectors to Describe Motion

1. Tangent Vector $\vec{T} = d\vec{r} / ds$
2. Normal Vector $\vec{N} = 1/\kappa \; d\vec{T}/ds$
3. Binormal Vector $\vec{B} = \vec{T} \times \vec{N}$

cross product



What does $d\vec{B}/ds$ tell us?

$$d\vec{B}/ds = d(\vec{T}\times\vec{N})/ds$$

$$= \underbrace{d\vec{T}/ds}_{} \times \vec{N} + \vec{T} \times d\vec{N}/ds$$

$d\vec{T}/ds = \kappa\vec{N}$

$$= \cancel{\kappa\vec{N} \times \vec{N}}^{\,0} + \vec{T} \times d\vec{N}/ds$$

$$= \vec{T} \times d\vec{N}/ds$$

So...

1) $d\vec{B}/ds$ orthogonal to $\vec{T}$
2) $d\vec{B}/ds$ orthogonal to $\vec{B}$

$\Rightarrow$ $d\vec{B}/ds$ parallel to $\vec{N}$

$\Rightarrow$ $-d\vec{B}/ds = \tau\vec{N}$ for some τ

$-d\vec{B}/ds \cdot \vec{N} = \tau\vec{N}\cdot\vec{N}$

$\tau = 0 \Rightarrow$ "no twisting"

Definition: torsion...

$$-d\vec{B}/ds \cdot \vec{N} = \tau$$

Note: κ or curvature shows how points "curve" in th plane defined by $\vec{T}$ & $\vec{N}$
τ or torsion shows how th plane "twists"

- ☐ Review of th formulas
- ☐ Formulas and example for Torsion and Curvature. Note: We are not covering Torsion so you can skip over it!

Tangential and Normal Components of Acceleration

- ☐ $\kappa = |d\vec{T}/ds|$, $\kappa = 1/|\vec{v}| \; |d\vec{T}/dt|$, $\vec{T} = \vec{v}/|\vec{v}|$
- ☐ $\vec{N} = 1/\kappa \; d\vec{T}/ds$, $\vec{N} = 1/|d\vec{T}/dt| \cdot d\vec{T}/dt$
- ☐ Radius of th Osculating Circle $= \rho = 1/\kappa$
- ☐ Center of th Osculating Circle: $\vec{r}(s) + \rho(s)\vec{N}$

We define $\vec{B} = \vec{T} \times \vec{N}$

- ☐ The Scalar Tangential and Normal Components of acceleration are:

$$a_T = d^2s/dt^2 = \frac{d|\vec{v}|}{dt}, \quad a_N = \kappa\left(ds/dt\right)^2 = \kappa(|\vec{v}|)^2 = |\vec{v}|^2/\rho$$

Consider $\vec{v} \times \vec{a} = d\vec{r}/dt \times \vec{a}$; $\vec{a} = d^2s/dt^2\, \vec{T} + \kappa\,(ds/dt)^2\, \vec{N}$

$= d\vec{r}/ds \cdot ds/dt \times \vec{a}$

$= (ds/dt)\,\vec{T} \times \vec{a}$

$= (ds/dt)\,\vec{T} \times \left[d^2s/dt^2\, \vec{T} + \kappa\,(ds/dt)^2\, \vec{N} \right]$

$= \kappa\,(ds/dt)^3\, \underbrace{\vec{T} \times \vec{N}}_{\vec{B}}$

$= \kappa\,(ds/dt)^3\, \vec{B}$

$\Rightarrow \; |\vec{v} \times \vec{a}| = \kappa\, |ds/dt|^3$

<u>Finally</u> $\quad \kappa = |\vec{v} \times \vec{a}| \,/\, |\vec{v}|^3$

We have $\vec{B} = \vec{T} \times \vec{N}$

$d\vec{B}/ds = \overset{0}{\cancel{d\vec{T}/ds \times \vec{N}}} + \vec{T} \times d\vec{N}/ds$, $\kappa\vec{N} = d\vec{T}/ds$

$\Rightarrow d\vec{B}/ds = \vec{T} \times d\vec{N}/ds$

- $d\vec{B}/ds$ is orthogonal to $\vec{T}$
- Since $\vec{B}$ has a constant magnitude, $\vec{B} \cdot d\vec{B}/ds = 0$.
  $\Rightarrow d\vec{B}/ds$ is orthogonal to $\vec{B}$

$\Rightarrow d\vec{B}/ds$ is parallel to $\vec{N}$

$\Rightarrow d\vec{B}/ds = -\tau\vec{N}$

$$\boxed{\Rightarrow \tau = -\,d\vec{B}/ds \cdot \vec{N}}$$

Torsion

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|}, \text{ if } \vec{v} \times \vec{a} \neq \vec{0}$$

<u>Example</u> : Consider $\vec{r}(t) = \langle \overset{x(t)}{a\cos t}, \overset{y(t)}{a\sin t}, \overset{z(t)}{bt} \rangle$, $a, b \geq 0$, $a^2 + b^2 \neq 0$.

$\vec{v}(t) = \langle -a\sin t, a\cos t, b \rangle$

$\vec{a}(t) = \langle -a\cos t, -a\sin t, 0 \rangle$

Note: $\tau = \dfrac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = a^2 b \,/\, \left(a\sqrt{a^2+b^2}\right)^2$

$\Rightarrow b/a^2 + b^2$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{vmatrix} = \hat{\imath}(ab\sin t) - \hat{\jmath}(ab\cos t) + k(a^2)$$

$= \langle ab\sin t, -ab\cos t, a^2 \rangle$

$|\vec{v}| = \sqrt{a^2\sin^2 t, a^2\cos^2 t + b^2}$

$= \sqrt{a^2 + b^2}$

$|\vec{v} \times \vec{a}| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4}$

$= \sqrt{a^2b^2 + a^4}$

$= a\sqrt{a^2 + b^2}$

$|\vec{v}| = (a^2 + b^2)^{3/2}$

$\kappa = |\vec{v} \times \vec{a}| / |\vec{v}|^3 = a/(a^2+b^2)^{3/2 - 1/2} = a/a^2 + b^2$

- ☐ Interpret the normal and tangential components of acceleration
- ☐ Compute the normal and tangential components of acceleration
- ☐ Use the Pythagorean identity to simplify computation of the normal component of acceleration

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Tangential and Normal Components of Acceleration

$\triangle$ $\vec{v} = |\vec{v}| \dfrac{\vec{v}}{|\vec{v}|} = |\vec{v}|\vec{T}$

$\triangle$ $\vec{N} = \dfrac{d\vec{T}/dt}{|d\vec{T}/dt|}$ $\quad$ $\triangle$ $K = \dfrac{1}{|\vec{v}|}\left|d\vec{T}/dt\right|$

$\triangle$ $\vec{a} = d\vec{v}/dt = d|\vec{v}|/dt\ \vec{T} + |\vec{v}|\ d\vec{T}/dt$

$= d|\vec{v}|/dt\ \vec{T} + |\vec{v}|\,|d\vec{T}/dt|\,\vec{N}$

$= \underbrace{d|\vec{v}|/dt}_{a_{\vec{T}}}\ \vec{T} + \underbrace{K|\vec{v}|^2}_{a_{\vec{N}}}\ \vec{N}$

Definition

For $a_{\vec{T}} = \dfrac{d|\vec{v}|}{dt}$, $a_{\vec{N}} = K|\vec{v}|^2$

$\vec{a} = a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N}$

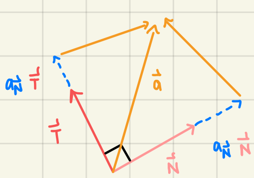
Example: Circle of radius 1: $\vec{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$

$\vec{v} = d\vec{r}/dt = -(\sin t)\hat{\imath} + (\cos t)\hat{\jmath}$

$\Rightarrow |\vec{v}| = 1 \qquad K = 1/1 = 1$

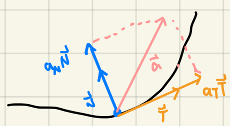
$a_{\vec{T}} = d|\vec{v}|/dt = 0$

$a_{\vec{N}} = K|\vec{v}|^2 = 1$



$|\vec{a}| = \sqrt{|a_{\vec{N}}|^2 + |a_{\vec{T}}|^2}$

- ☐ Formulas for the Tangential and Normal component of Acceleration
- ☐ Example of the Tangential and Normal component of Acceleration



$\vec{a} = a_T\vec{T} + a_N\vec{N}$

Tangential and Normal Components of Acceleration

Recall: $\vec{v} = d\vec{r}/dt = d\vec{r}/ds \cdot ds/dt \Rightarrow \vec{v} = \vec{T} \cdot ds/dt$

$\Rightarrow \vec{a} = d\vec{v}/dt = \dfrac{d}{dt}(\vec{T} \cdot ds/dt) = d^2s/dt^2\,\vec{T} + ds/dt\ d\vec{T}/ds$, $d\vec{T}/ds = K\vec{N}$

$= d^2s/dt^2\,\vec{T} + ds/dt\ d\vec{T}/ds \cdot ds/dt$

$= \underbrace{d^2s/dt^2}_{a_T}\,\vec{T} + \underbrace{K(ds/dt)^2\vec{N}}_{a_N}$

$\vec{a} = a_T\vec{T} + a_N\vec{N}$ where

$a_T = d^2s/dt^2 = d|\vec{v}|/dt$ ←

$a_N = K(ds/dt)^2 = K|\vec{v}|^2$ ←

Example: Write the acceleration of the motion describe by...

$\vec{r}(t) = \langle \cos t + t\sin t,\ \sin t - t\cos t \rangle,\ t > 0$

First $\vec{v}(t) = d\vec{r}/dt$

$\vec{v} = d\vec{r}/dt = \langle \cancel{-\sin t + \sin t} + t\cos t,\ \cancel{\cos t - \cos t} + t\sin t \rangle$ 0 0

$\Rightarrow \vec{v}(t) = \langle t\cos t, t\sin t \rangle$

$\Rightarrow |\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = t$

$= t$

Now since $a_T = d|\vec{v}|/dt$

$= d(t)/dt$

$= 1$

$\vec{a} = a_T\vec{T} + a_N\vec{N}$

$\vec{a}\cdot\vec{a} = a_T^2 + a_N^2$

$|\vec{a}|^2 = a_T^2 + a_N^2$

$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$

$\vec{a}(t) = \langle \cos t - t\sin t,\ \sin t + t\cos t \rangle$

$|\vec{a}(t)|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2$

$= \cos^2 t + t^2\sin^2 t - 2t\cancel{\cos t}\sin t + t\sin^2 t + t^2\cos^2 t + 2t\cancel{\cos t}\sin t$

$\Rightarrow |\vec{a}(t)|^2 = 1 + t^2$

Using this, we write $a_N = \sqrt{|\vec{a}|^2 - a_T^2}$

$= (1+t^2) - 1$

$= t$

We have $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$\boxed{\Rightarrow \vec{a}(t) = \vec{T} + t\vec{N}}$

Quiz on 13.5 100%

1. $r(t) = \cos t\, i + \sin t\, j + 0k$ compute B @ $t=0$,

$\vec{B} = \vec{T} \times \vec{N}$

$\langle \cos t, \sin t, 0 \rangle$

$\langle -\sin t, \cos t, 0 \rangle = \vec{v}$

$= \vec{T}$

$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 0^2}$

$= 1$

$\vec{N} = \langle -\cos t, -\sin t, 0 \rangle$

$$\begin{vmatrix} i & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$(\sin^2 t - (-\cos^2 t))\hat{k}$

$1k$

$\vec{B} = \langle 0, 0, 1 \rangle$

2. $\vec{r}(t) = \langle 1, 2t, t^2/2 \rangle$ compute $a_T(1)$ $t=1$

$a_T = d|\vec{v}|/dt$ $\langle 0, 2, t \rangle = \sqrt{0^2 + 2^2 + t^2}$

$= \sqrt{4+t^2}$

$= t/\sqrt{4+t^2}$

$= 1/\sqrt{5}$

$= 0.44721$

≈ 0.45

3. $\vec{r}(t) = (1+3t)i + (t-2)j - 3tk$ a_N

$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$

$= \sqrt{0^2 - 0^2}$

$= \sqrt{0}$

$\langle 1+3t, t-2, -3t \rangle$ $\vec{r}(t)$

$\langle 3, 1, -3 \rangle$ $\vec{v}(t)$

$\langle 0, 0, 0 \rangle$ $\vec{a}(t)$

$|\vec{v}| = \sqrt{3^2 + 1^2 + 3^2}$

$= \sqrt{19}$

$a_T = 0$

$|\vec{a}(t)| = \sqrt{0}$

$= 0$

4. $\vec{v}(t) = i + tj + t^2 k$ $\vec{a}(t) = j + 2tk$

torsion at $t=0$

$$\tau = \frac{\begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = 2$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{vmatrix}$$

$\hat{i}(2t^2 - t^2) = \hat{i}t^2$

$-\hat{j}(2t) = -\hat{j}(2t)$

$\hat{k}(1) = \hat{k}$

$\sqrt{t^4 + 4t^2 + 1}$