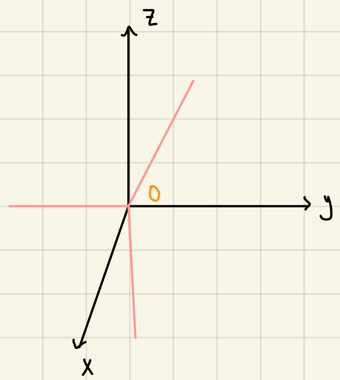
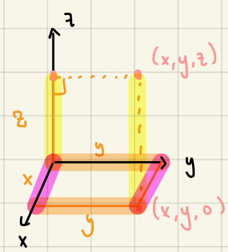
12.1 – 3D Coordinate Systems

- Introduction to 3D Coordinate System
- Interpret some basic algebraic equations geometrically
- Distance formula and equation of a Sphere



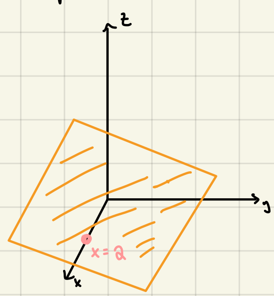
There are basically 8 different cells called **octants**.

ex.

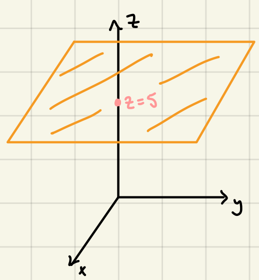


Geometric Interpretations

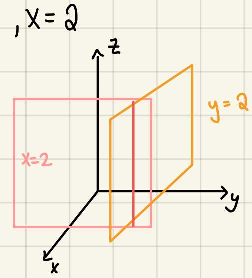
a) $x = 2$



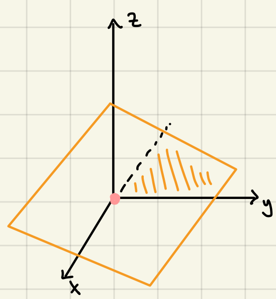
b) $z = 5$



f) $y=2$, $x=2$



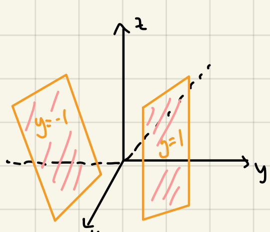
c) $z=0,\ x \leq 0,\ y \geq 0$



d) $x \geq 0,\ y \geq 0,\ z \geq 0$

First Octant!

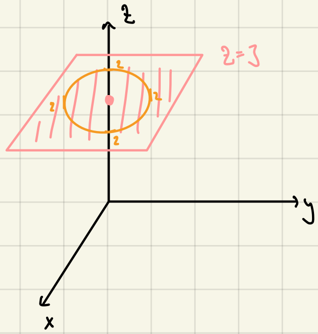
e) $-1 \leq y \leq 1$



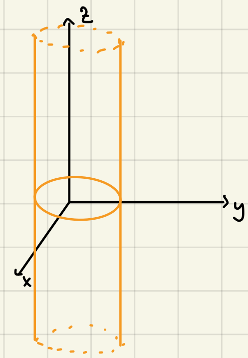
slab between the planes $y=-1$ and $y=1$.

Graph of a Set of Equations

a) Point (x, y, z) where $x^2+y^2=4$ and $z=3$.



∘ $x^2+y^2=4$



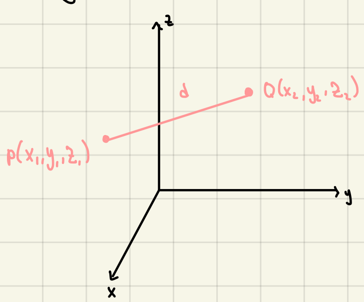
Distance and Sphere in Space

∘ The distance between the points (x_1, y_1) and (x_2, y_2) is ...

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Set of all points in 2D whose distance from the origin is a constant (fixed) is called a circle.

a) $x^2+y^2+z^2+3x-4z+1=0$



$$(x^2+3x)+y^2+(z^2-4z)+1=0$$

$$\Rightarrow (x^2+3x+9/4)+y^2+(z^2-4z+4)+1=9/4+4$$

$$\Rightarrow (x+3/2)^2+(y-0)^2+(z-2)^2=(9+16)/4-1$$

$$\Rightarrow (x+3/2)^2+(y-0)^2+(z-2)^2=21/4$$

$$=\left(\sqrt{21}/2\right)^2$$

∴ Centre is $(-3/2, 0, 2)$

radius $=\sqrt{21}/2$

Distance between P and Q is...

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

The equation of a sphere centred at the origin is...

$$x^2+y^2+z^2=r^2$$

Or

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=r^2$$

Thus... Distance $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

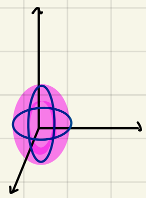
Equation of a sphere centred at (x_0, y_0, z_0) having radius r is...

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

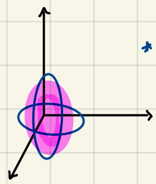
$$x^2+y^2+z^2=r^2$$

Geometric Interpretations

a) $x^2 + y^2 + z^2 \leq 4$

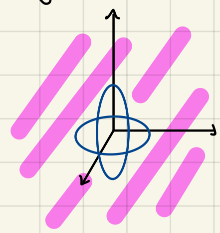


b) $x^2 + y^2 + z^2 < 4$



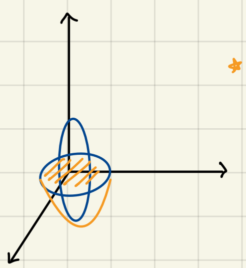
* Everything but th boundry of th sphere.

c) $x^2 + y^2 + z^2 > 4$



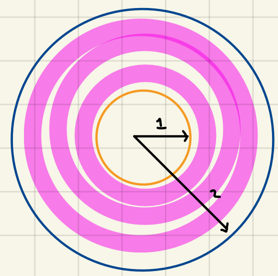
d) $x^2 + y^2 + z^2 = 4$, $z \leq 0$

spherical shell



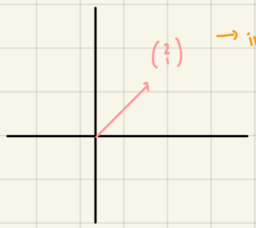
* just th shell, nothing inside.

e) $1 \leq x^2 + y^2 + z^2 \leq 4$



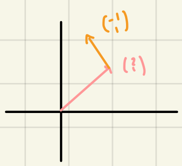
12.2 - Vectors

- Describe vectors both algebraically and geometrically
- Define addition and scalar multiplication of vectors algebraically and geometrically
- Prove the parallelogram law

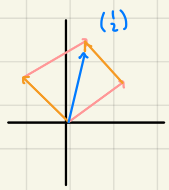


→ instruction 2 to the right
1 up.

- Scalar multiplication: $c\binom{a}{b} = \binom{ca}{cb}$



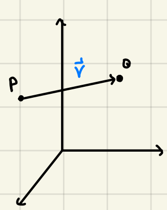
- Vector addition ("Tip-to-Tail"):



- Basic definitions and notations for vectors
- Vector algebraic operations and properties of vectors operations
- Applications

A quantity such as force, displacement, and velocity is a vector.

Represent



$\vec{V}$, $\overrightarrow{PQ}$, V

P = Initial Point

Q = Terminal Point

- Suppose $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the vector $\vec{v}\,(\overrightarrow{PQ})$ is represented as...

$$\vec{V} = \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Note: This ↗ is called the component form of $\vec{v}$.

$$\vec{V} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

- Magnitude or Length of a Vector:

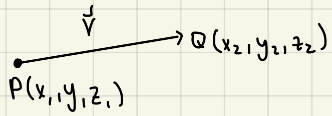
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

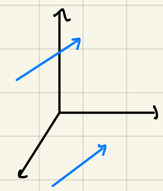
Note: The only vector that has a length of zero is the zero vector.
$\vec{0} = \langle 0,0,0\rangle$

Note: Zero vector does not have any direction

Summary

- $\vec{v} = \langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$
- $|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
- $\vec{0} = \langle 0,0,0\rangle$, No direction
- Two vectors are equal if they have the same length and direction.





Components and magnitude of a vector

a) $P(-3,4,1)$ and $Q(-5,2,2)$

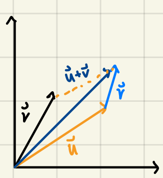
$\vec{v} = \vec{PQ} = \langle -5-(-3), 2-4, 2-1\rangle$
$= \langle -2,-2,1\rangle$; component.

$|\vec{v}| = \sqrt{(-2)^2+(-2)^2+(1)^2}$
$= 3$

Vector Algebra Operations

- Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ be two vectors...

$\vec{u}+\vec{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3\rangle$



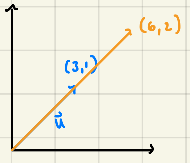
- Scalar multiplication: Let k be a scalar

$|k\vec{u}| = |k||\vec{u}|$

$k\vec{u} = \langle ku_1, ku_2, ku_3\rangle$

$|k\vec{u}| = \sqrt{(ku_1)^2+(ku_2)^2+(ku_3)^2}$

$= |k|\sqrt{u_1^2+u_2^2+u_3^2}$
$= |k||\vec{v}|$



$\vec{u} = \langle 3,1\rangle$
$2\vec{v} = \langle 6,2\rangle$

$|2\vec{u}| = \sqrt{36+4} = 2\sqrt{10}$
$|\vec{u}| = \sqrt{9+1} = \sqrt{10}$
$|2||\vec{u}| = 2\sqrt{10}$ ✓

- Subtraction: $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$
$= \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$

Components of a vector

c) $u = \langle -1, 3, 1 \rangle$ and $v = \langle 4, 7, 0 \rangle$

i) $2u + 3v$ $\quad 2\vec{u} = \langle -2, 6, 2 \rangle$ $\quad = \langle 10, 27, 2 \rangle$
$3\vec{v} = \langle 12, 21, 0 \rangle$

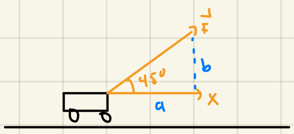
ii) $u - v$ $\quad = \vec{u} + (-\vec{v})$ $\quad -\vec{v} = \langle -4, -7, 0 \rangle$
$= \langle -1-4, 3-7, 1+0 \rangle$
$= \langle -5, -4, 1 \rangle$

iii) $\left|\frac{1}{2}u\right|$ $\quad \frac{1}{2}\vec{u} = \langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \rangle$

$\left|\frac{1}{2}\vec{u}\right| = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}$
$= \sqrt{11}/2$

Components and magnitude of a vector

b) A small cart is being pulled along a smooth horizontal floor w/ a 20-lb force f making a 45° angle to the floor. What is the effective force moving the cart forward?



If $\vec{F} = \langle a, b \rangle$
$a = |\vec{F}| \cos 45°$
$b = |\vec{F}| \sin 45°$

$\cos 45° = a / |\vec{F}|$
$\sin 45° = b / |\vec{F}|$

Here $|\vec{F}| = 20$
$a = 20 \cos 45°$
$= 20\sqrt{2}/2$

$\therefore 10\sqrt{2} = a$

Effective force $= \langle 10\sqrt{2}, 0 \rangle$
$\approx \langle 14.14, 0 \rangle$

Properties of Vector Operations

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be three vectors and a, b be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $0\vec{u} = \vec{0}$
- $1\vec{u} = \vec{u}$
- $(ab)\vec{u} = a(b\vec{u})$
- $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

Unit Vectors

A vector $\vec{v}$ of Length 1 is called a unit vector.

Standard unit Vectors:

$$i = \langle 1,0,0 \rangle$$
$$j = \langle 0,1,0 \rangle$$
$$k = \langle 0,0,1 \rangle$$

Notation: $\hat{\imath} = \langle 1,0,0 \rangle, \hat{\jmath} = \langle 0,1,0 \rangle, \hat{k} = \langle 0,0,1 \rangle$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\langle 1,0,0 \rangle + v_2\langle 0,1,0 \rangle + v_3\langle 0,0,1 \rangle$$
$$= v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

← k^{th} component

↑ j^{th} component

↑ i^{th} Component

We can convert a given vector into a unit vector by simply dividing it w/ its own magnitude.

i.e. $\check{v} = \vec{v}/|\vec{v}|$; $|\vec{v}| \neq 0$.

$$|\check{v}| = \left|\vec{v}/|\vec{v}|\right| = \frac{1}{|\vec{v}|}|\vec{v}| = 1$$

d) $P_1(1,0,1)$ and $P_2(3,2,0)$

$$\vec{u} = \overrightarrow{P_1P_2} = \langle 3-1, 2-0, 0-1 \rangle$$
$$= \langle 2,2,-1 \rangle$$

$$|\vec{u}| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\check{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u}}{3} = \frac{1}{3}\langle 2,2,-1 \rangle$$
$$= \langle 2/3, 2/3, -1/3 \rangle, \quad \boxed{|\check{u}| = 1}$$

e) If $v = 3i - 4j$ is a velocity vector, express v as a product of its speed times its direction of motion.

Speed : $|\vec{v}| = \sqrt{9+16} = 5 \longrightarrow$ speed

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}\langle 3, -4\rangle = \langle 3/5, -4/5\rangle$$

$\vec{v} =$ (speed)(direction of motion)

$= 5\,\langle 3/5, -4/5\rangle$

f) A force of 6 newtons is applied in the direction of the vector $v = 2i + 2j - 1k$. Express the force F as a product of its magnitude and direction.

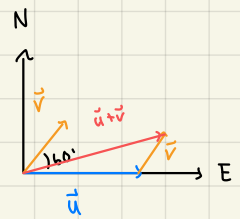
We can't simply do: $\vec{F} = 6\langle 2, 2, -1\rangle$.
$|\vec{F}| \neq 6$.

$|\vec{v}| = \sqrt{4+4+1} = 3$

$\hat{v} = \vec{v}/|\vec{v}| = \langle 2/3, 2/3, -1/3\rangle$

Now; $\vec{F} = 6\langle 2/3, 2/3, -1/3\rangle$, $|\vec{F}| = 6$

g) A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of wind, acquires a new ground speed and direction. What are they?



$\vec{u} = \langle 500, 0\rangle$

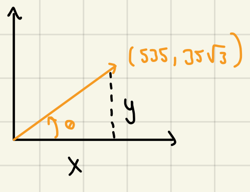
$\vec{v} = \langle 70\cos 60°, 70\sin 60°\rangle$

$= \langle 35, 35\sqrt{3}\rangle$

$\vec{u}+\vec{v} = \langle 535, 35\sqrt{3}\rangle$

$|\vec{u}+\vec{v}| = \sqrt{(535)^2 + (35\sqrt{3})^2}$

≈ 538 mph



$\theta = \tan^{-1}\left(\frac{35\sqrt{3}}{535}\right)$

$\approx 6.5°$

Quiz on 12.1 / 12.2

100%

✓ 1 Compute $\begin{pmatrix}3\\2\end{pmatrix} - 2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix} \longrightarrow \begin{pmatrix}1\\4\end{pmatrix}$

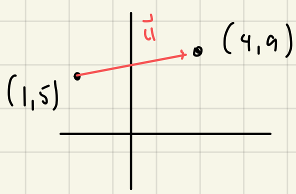
✓ 2. $x^2 - 2x + y^2 - 2y + z^2 - 4z - 20 = 0$

$$(x^2-2x) = (x-1)^2 - 1$$
$$(y^2-2y) = (y-1)^2 - 1$$
$$(z^2-4z) = (z-2)^2 - 4$$

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = 20+1+1+4$$
$$= 26 \quad = (\sqrt{26})^2$$

Centre is $(1,1,2)$
radius = $\sqrt{26}$

x 3. $\vec{u}$ $A=(1,5)$ $B=(4,9)$ y-coord $-\vec{u}$?



$\langle 4-1, 9-5 \rangle$
$\langle 3, 4 \rangle$

opposite $\langle 1-4, 5-9 \rangle$
$\langle -3, -4 \rangle$

unit vector

$|\vec{u}| = \sqrt{9+16} = 5$

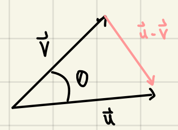
$\frac{1}{5}\langle -3, -4 \rangle$
$\langle -3/5, -4/5 \rangle \approx -0.8$ ✓

12.3 — Dot Product (pg 728-730 of section 12.3)

Dot products are an operation that combines two vectors into a scalar. They turn out to be fundamentally connected to the geometry of $\mathbb{R}^n$ and useful for many things.

- ☐ Compute the dot product of two vectors
- ☐ Compute the angle between two nonzero vectors
- ☐ Describe the geometric properties of the dot product

Angle Between Vectors & Dot Product



Cosine Law:

$(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2$ ★

$$||\vec{u}-\vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||\,||\vec{v}||\cos(\theta)$$

$||\vec{u}||^2 = u_1^2+u_2^2+u_3^2$ $\quad$ $||\vec{v}||^2 = v_1^2+v_2^2+v_3^2$

★ $= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2) - 2u_1v_1 - 2u_2v_2 - 2u_3v_3$

$= -2\,||\vec{u}||\,||\vec{v}||\cos(\theta)$

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{||\vec{u}||\,||\vec{v}||}\right)$$

The angle between nonzero vectors $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$ is given by...

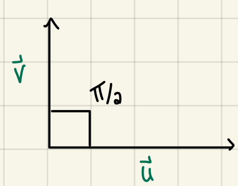
For $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$, the <u>dot product</u> is:

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

($\vec{u}\cdot\vec{v}$: two vectors; $u_1v_1 + u_2v_2 + u_3v_3$: scalar)

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}\right)$$

Ex.



$\pi/2 = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}\right)$

$\longrightarrow \cos(\pi/2) = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||} = 0$ $\quad \Rightarrow \vec{u}\cdot\vec{v}=0$

Vectors $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$ are <u>orthogonal</u> if $\vec{u}\cdot\vec{v}=0$

- ☐ Examples on dot product of two vectors
- ☐ Examples on how to find th angle between two nonzero Vectors
- ☐ Algebraic and Geometric properties of th dot product
- ☐ Determining th orthogonality of two vectors using th dot product

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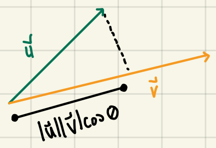
The Dot Product



The dot product of Vectors $\vec{u}$ and $\vec{v}$

$$\vec{u}=\langle u_1,v_2,u_3\rangle \quad , \quad \vec{v}=\langle v_1,v_2,v_3\rangle$$

is th scalar quantity...

$$\vec{u}\cdot\vec{v} = \boxed{u_1v_1+u_2v_2+u_3v_3} \rightarrow \text{scalar}$$

(*) Notice $\vec{u}\cdot\vec{u}=\langle u_1,u_2,u_3\rangle\cdot\langle u_1,u_2,u_3\rangle$
$= u_1^2+u_2^2+u_3^2$

$\Rightarrow \vec{u}\cdot\vec{u} = u_1^2+u_2^2+u_3^2$

$|\vec{u}| = \sqrt{u_1^2+u_2^2+u_3^2}$
$|\vec{u}|^2 = u_1^2+u_2^2+u_3^2$

$$\vec{u}\cdot\vec{u} = |\vec{u}|^2$$

Example 1

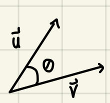
a) $u=\langle 1,-2,-1\rangle,\ v=\langle -6,2,-3\rangle$

$$\begin{aligned}\vec{u}\cdot\vec{v} &= \langle 1,-2,-1\rangle\cdot\langle -6,2,-3\rangle\\ &= (1)(-6)+(-2)(2)+(-1)(-3)\\ &= -6-4+3\\ &= -7\end{aligned}$$

b) $u=\frac{1}{2}i+3j+1k,\ v=4i-1j+2k$

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (\tfrac{1}{2})(4)+(3)(-1)+(1)(2)\\ &= 2-3+2\\ &= 1\end{aligned}$$

Angle Between Two Vectors



Theorem:

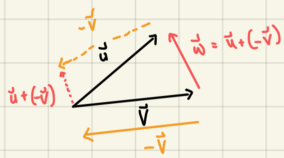
The angle θ between two vectors (non-zero) $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$ is:

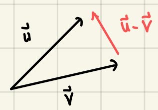
$$\theta = \cos^{-1}\left[\frac{u_1v_1+u_2v_2+u_3v_3}{|\vec{u}|\cdot|\vec{v}|}\right]$$

↓

or, we can write $\theta = \cos^{-1}\left[\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right]$ $\Rightarrow$ $\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$





$\vec{u}\cdot\vec{u} = |\vec{u}|^2$

Example 2

$u = 1i - 2j - 2k$ and $u = 6i + 3j + 2k$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

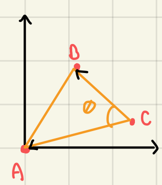
$$= \cos^{-1}\left(\frac{6-6-4}{\sqrt{1+4+4}\cdot\sqrt{36+9+4}}\right)$$

$$= \cos^{-1}\left(\frac{-4}{3\cdot 7}\right) = \cos^{-1}\left(\frac{-4}{21}\right)$$

≈ 1.8 radians (101 degrees)

Example 3

Find the angle θ in the triangle ABC determined by the vertices $A=(0,0)$, $B=(3,5)$, and $C=(5,2)$.



$\vec{CA} = \langle -5, 2\rangle$ $\quad |\vec{CA}| = \sqrt{25+4} = \sqrt{29}$

$\vec{CB} = \langle -2, 3\rangle$ $\quad |\vec{CB}| = \sqrt{4+9} = \sqrt{13}$

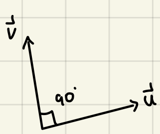
$\vec{CA}\cdot\vec{CB} = -10 + 6 = -4$ ($+10 - 6 = 4$)

$$\theta = \cos^{-1}\left(\frac{\vec{CA}\cdot\vec{CB}}{|\vec{CA}||\vec{CB}|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\cdot\sqrt{13}}\right)$$

$\approx 78.1^\circ$ (1.36 radians)

Orthogonal Vectors

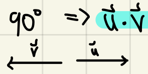
$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$



If the angle between vectors is 90° $\Rightarrow$ $\vec{u}\cdot\vec{v} = 0$

If $\theta = 180^\circ$, $\vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$

If $\theta = 0^\circ$, $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$



Example 4

a) $u = \langle 3, -2 \rangle$, $v = \langle 4, 6 \rangle$

$$\vec{u}\cdot\vec{v} = (2)(4) + (-2)(6) = 12 - 12 = 0$$

$\therefore$ $\vec{u}$ is indeed orthogonal to $\vec{v}$. $(\vec{u} \perp \vec{v})$

b) $u = 3i - 2j + 1k$, $v = 2j + 4k$

$$\vec{u}\cdot\vec{v} = (3)(0) + (-2)(2) + (1)(4) = 0$$

$\therefore$ $\vec{u} \perp \vec{v}$.

c) $0 = \langle 0, 0, 0 \rangle$ is orthogonal to every vector u.

Suppose $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$$\Rightarrow \vec{u}\cdot\vec{0} = (u_1)(0) + (u_2)(0) + (u_3)(0) = 0$$

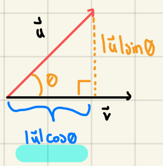
Further Properties of the Dot Product

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be three vectors and c be any scalar.

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ Commutative $\vec{u} = \langle u_1, u_2 \rangle, \vec{v} = \langle v_1, v_2 \rangle$ / $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \vec{v}\cdot\vec{u}$
- $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ Distribution Property
- $\vec{0}\cdot\vec{u} = 0$
- $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$ Scalar multiplication
- $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

- Computing the scalar component of one vector along another vector
- Computing the vector projection of one vector onto the another vector
- Decomposition of a vector using projections
- Application of projections in computing Work

Vector Projections



$\vec{u} = \langle |\vec{u}|\cos\theta, |\vec{u}|\sin\theta \rangle$

We know that $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$. The scalar component of $\vec{u}$ along the vector $\vec{v}$ is

$$|\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{1}{|\vec{v}|}(\vec{u}\cdot\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

$$|\vec{u}|\cos\theta = \vec{u}\cdot\frac{\vec{v}}{|\vec{v}|} = \vec{u}\cdot(\text{unit vector in the direction of } \vec{v})$$

$$= |\vec{u}|\cdot\left|\frac{\vec{v}}{|\vec{v}|}\right|\cos\theta$$

$=1$

What we want now is a vector that goes in the direction of $\vec{v}$ and has a magnitude of $|\vec{u}|\cos\theta$.

We want the magnitude to be $|\vec{u}|\cos\theta$ and we want the direction to be the direction of $\vec{v}$.

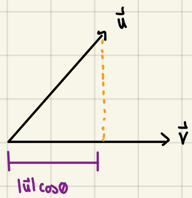
The required vector is $|\vec{u}|\cos\left(\frac{\vec{v}}{|\vec{v}|}\right)$.

Projection of $\vec{u}$ onto $\vec{v}$ = $\text{Proj}_{\vec{v}}\,\vec{u} = |\vec{u}|\cos\theta\left(\frac{\vec{v}}{|\vec{v}|}\right)$

Notice: $\text{Proj}_{\vec{v}}\,\vec{u} = \left(\frac{|\vec{u}|\cos\theta}{|\vec{v}|}\right)\vec{v}$

$= \left(\frac{|\vec{u}|\cos\theta\,|\vec{v}|}{|\vec{v}|\cdot|\vec{v}|}\right)\vec{v}$

$= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$



Scalar component of $\vec{u}$ in the direction of $\vec{v}$ = $|\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

Example 5

projection of $u = 6i + 3j + 2k$ onto $v = 1i - 2j - 2k$ and scalar component of u in the direction of v.

$$\text{Proj}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{6-6-4}{(\sqrt{1+4+4})^2}\right)\langle 1,-2,-2\rangle = \left(\frac{-4}{9}\right)\langle 1,-2,-2\rangle$$

$$= \langle (-4/9)(1), (-4/9)(-2), (-4/9)(-2)\rangle$$

$$= \langle -4/9, 8/9, 8/9\rangle$$

$$= \text{Proj}_{\vec{v}}\,\vec{u}$$

Scalar component of $\vec{u}$ in the direction of $\vec{v}$

$$= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = -4/\sqrt{9} = -4/3$$

Example 7

Verify that the vector $u - Proj_{\vec{v}} u$ is orthogonal to the projection vector $Proj_{\vec{v}} u$.

Same direction as $\vec{v}$.

Check

$$\left(\vec{u} - Proj_{\vec{v}}\vec{u}\right)\cdot\vec{v} = \vec{u}\cdot\vec{v} - Proj_{\vec{v}}\vec{u}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\cdot\vec{v}$$

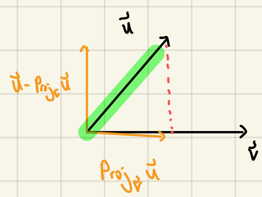
$$= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)|\vec{v}|^2$$

$$= \vec{u}\cdot\vec{v} - (\vec{u}\cdot\vec{v})$$

$$= 0 \; . \; ■$$

$\therefore$ $\vec{u} - Proj_{\vec{v}}\vec{u}$ and $Proj_{\vec{v}}\vec{u}$ are orthogonal.

$$\vec{u} = Proj_{\vec{v}}\vec{u} + \left[\vec{u} - Proj_{\vec{v}}\vec{u}\right]$$



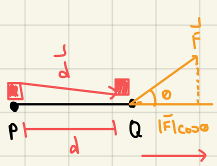
$$Proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

Scalar Component of $\vec{u}$ along $\vec{v}$ $= \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

$\vec{u} = Proj_{\vec{v}}\vec{u} + \vec{u} - Proj_{\vec{v}}\vec{u}$

Example 8

If $|F| = 40N$ (newtons), $|D| = 3m$, and $\theta = 60°$, find the work done by F in acting from a point P to a point Q.



Work done $= (|\vec{F}|\cos\theta)(\text{displacement})$
$= (|\vec{F}|\cos\theta)|\vec{PQ}|$
$= (|\vec{F}|\cos\theta)|\vec{d}|$
$= \vec{F}\cdot\vec{d}$

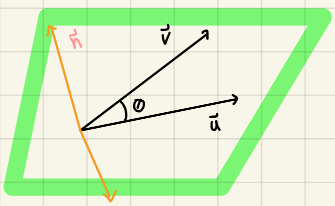
Work done $= \vec{F}\cdot\vec{d}$

Work $= |\vec{F}|\cdot|\vec{d}|\cdot\cos\theta$
$= (40)(3)\cdot\cos(60°)$
$= 60$ J (joules).

12.4 — Cross Products (pg. 736-738 of section 12.4)

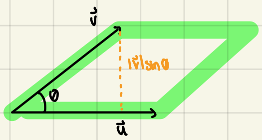
- ☐ Compute the cross product of two (three dimensional) vectors
- ☐ Compute the area of the parallelogram formed by two vectors
- ☐ Find a vector orthogonal to two other vectors
- ☐ Describe the geometric properties of the cross product

Cross Product



- normal to $\vec{u}$ and $\vec{v}$!

Set $\vec{n}$ the unit normal to $\vec{u}$ and $\vec{v}$ obeying Right Hand Rule



- Two vectors form a parallelogram
- Area of Parallelogram: base x height $= |\vec{u}||\vec{v}|\sin(\theta)$

Geometric Cross Product:

$$\vec{u} \times \vec{v} = \left(|\vec{u}||\vec{v}|\sin(\theta)\right)\vec{n}$$

$|\vec{u}||\vec{v}|\sin(\theta)$: area of parallelogram

$\vec{n}$ ← unit normal vector obeying R.H.R

Algebraic Cross Product:

For $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$$u \times v = (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k$$

Determinant Cross Product:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- ☐ Definition, properties and examples on Cross product of two vectors
- ☐ Using Cross product to compute the area of a parallelogram
- ☐ Using Cross product to compute the volume of a parallelogram

---

Cross Product

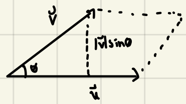
The cross product of two vectors $\vec{u}$ and $\vec{v}$ is the given by:

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\,\hat{n}$$

where $\hat{n}$ is the unit vector that perpendicular to the plane that contains both $\vec{u}$ and $\vec{v}$.

- Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = 0$.

Area of a Parallelogram:



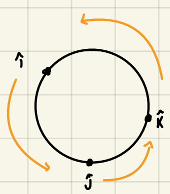
$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\,\hat{n}$$

$\Rightarrow \vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta$
$= (\text{base height})(\text{height})$
$= \text{Area of the parallelogram}$

Properties

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any given vectors and r and s are scalars, then:

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
4. $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$
5. $\vec{0} \times \vec{u} = \vec{0}$ (def.)
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ generally $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$



$\hat{i} \times \hat{j} = \hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

Determinant formula for $\vec{u} \times \vec{v}$:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\downarrow$

Example 1

Find $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$, $\vec{u} = 2i + 1j + 1K$ and $\vec{v} = -4i + 3j + 1K$

Note $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \hat{i}(1-3) - \hat{j}(2+4) + \hat{k}(6+4)$$
$$= -2\hat{i} - 6\hat{j} + 10\hat{k}$$

$\vec{v}\times\vec{u} = 2i + 6\hat{j} - 10\hat{k}$ because! $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

Example 2

Vector Perpendicular to the plane $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

Let's find the vectors:

$\vec{PR} = \langle -2, 2, 2\rangle$
$\vec{PQ} = \langle 1, 2, -1\rangle$

Now, $\vec{PR}\times\vec{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = \hat{i}(-2-4) - \hat{j}(2-2) + \hat{k}(-4-2)$
$= -6\hat{i} + 0\hat{j} - 6\hat{k}$

The required vector is $-6\hat{i} - 6\hat{k} = \langle -6, 0, -6\rangle$

Example 3

Find area of the triangle $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

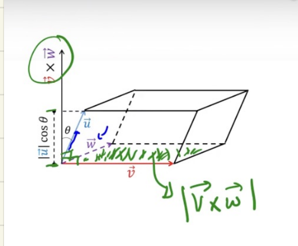
from the last example, we had $\vec{PQ}\times\vec{PR} = \langle 6, 0, 6\rangle$

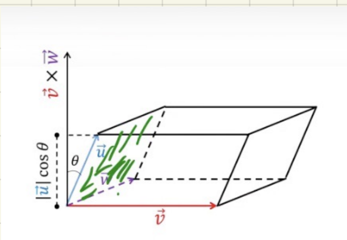
$|\vec{PQ}\times\vec{PR}| = \sqrt{36+36} = \sqrt{72}$; Area of the parrallelogram determined by $\vec{PQ}$ and $\vec{PR}$.

Area of the required triangle $= \frac{1}{2}\,|\vec{PQ}\times\vec{PR}|$
$= \sqrt{72}/2$
$= \sqrt{4\times 18}/2$
$= \sqrt{18} = 3\sqrt{2}$

Triple Scalar or Box Product

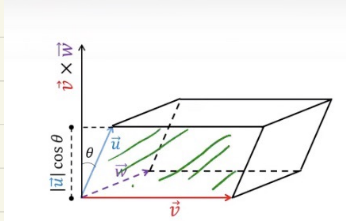
Volume $= |\vec{v}\times\vec{w}|\cdot|\vec{u}|\cos\theta$
$= |(\vec{v}\times\vec{w})\cdot\vec{u}|$
$= |\vec{u}\cdot(\vec{v}\times\vec{w})|$





$|\vec{w}\times\vec{u}||\vec{v}|\cos\theta$
$= |(\vec{w}\times\vec{u})\cdot\vec{v}|$
$= \vec{v}\cdot(\vec{w}\times\vec{u})$

$$|\vec{v}\times\vec{u}||\vec{u}|\cos\theta|$$
$$= |(\vec{v}\times\vec{u})\cdot\vec{w}|$$
$$= \vec{u}\cdot(\vec{v}\times\vec{u})$$



$$\text{Volume} = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$
$$= |\vec{v}\cdot(\vec{w}\times\vec{u})|$$
$$= |\vec{w}\cdot(\vec{v}\times\vec{u})|$$

$\vec{u}$, $\vec{v}$, and $\vec{w}$

$$(\vec{u}\times\vec{v})\,\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 6

Volume of the box: $u = 1i + 2j - 1k$, $v = -2i + 3k$, and $w = 7j - 4k$.

★ determinant

$$\text{Volume} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = 1(0-21) - 2(8-0) - 1(-14-0)$$
$$= -21 - 16 + 14$$
$$= -23$$

$$= 23 \text{ units}^3 = |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

$$\text{Area} = |\vec{u}\times\vec{v}|$$
$$\text{Volume} = |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

Quiz on 12.3/12.4

100%

✓ 1. Angle $u = \langle 2,2,0 \rangle$ and $\langle 2,-1,1 \rangle$ (10th of a radian)

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$$= \cos^{-1}\left(\frac{4-2+0}{\sqrt{4+4}+\sqrt{4+1+1}}\right) \longrightarrow 1.18$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{8}+\sqrt{6}}\right) \left(\frac{2}{\sqrt{8}\cdot\sqrt{6}}\right) = 1.277953555 \longrightarrow 1.3$$

or 73.22°

$\approx 1.18214\ldots$ or 67.73°

✓ 2. Volume box

$\langle 1,1,4 \rangle$, $\langle 2,1,3 \rangle$, and $\langle -4,3,2 \rangle$ (integer)

$$\text{Volume} = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 1 & 3 \\ -4 & 3 & 2 \end{vmatrix} = 1(2-9)-1(4+12)+4(6+4) = -7-16+40 = 17$$

$\longrightarrow 17$

✓ 3. z-coordinate normal vector (integer)

$P(-2,1,5)$, $Q(4,9,-3)$, and $R(3,-1,0)$

$\vec{PQ} \times \vec{PR}$.

$\vec{PQ} = \langle 4+2, 9-1, -3-5 \rangle$
$\langle 6, 8, -8 \rangle$

$\vec{PR} = \langle 3+2, -1-1, 0-5 \rangle$
$\langle 5, -2, -5 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 8 & -8 \\ 5 & -2 & -5 \end{vmatrix} = \hat{i}(-40-(16)) - \hat{j}(-30-(-40)) + \hat{k}(-12-(40))$$

$-56\hat{i} - 10\hat{j} - 52\hat{k}$

$\longrightarrow -52 \longrightarrow 52$

Tutorial #1 w/ Dr. Daniel Moghbel

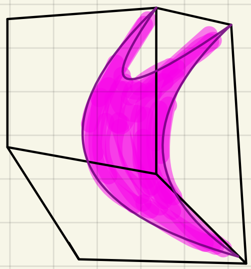
1. Solution Steps:

- Given point P and any point Q in the y,z-plane (i.e., Q(0,y,z)), express equidistance property in terms of third point R(x,y,z):

$$|\overrightarrow{PR}| = |\overrightarrow{QR}| \quad \text{— i.e.,} \quad ||\overrightarrow{PR}|| = ||\overrightarrow{QR}||$$

- Simplify equation from step 1:

$$(x-1)^2 + y^2 + z^2 = x^2 \iff x = \tfrac{1}{2}(1 + y^2 + z^2)$$

b) 

(Infinite Paraboloid)

4. Solution Steps:

- Let $\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}$ where $\vec{u}_{\parallel} \parallel \vec{w}$ and $\vec{u}_{\perp} \perp \vec{w}$.

- Since $\vec{u} = \vec{u} - \text{proj}_{\vec{w}}\vec{u} + \text{proj}_{\vec{w}}\vec{u}$, we take

$$\vec{u}_{\parallel} = \text{proj}_{\vec{w}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \left(\frac{-10-2+3}{4+4+1}\right)[-2,2,1]$$
$$= -[-2,2,1]$$
$$= [2,-2,-1]$$

and

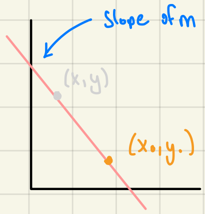
$$\vec{u}_{\perp} = \vec{u} - \text{proj}_{\vec{w}}\vec{u} = [5,-1,3] - [2,-2,-1]$$
$$= [3,1,4]$$

12.5 — Lines & Planes

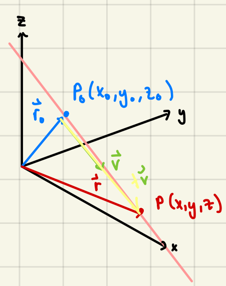
- [ ] Write the vector equation of a line given a point on the line and the direction of the line.

Vector Equation of Lines

- Equation of Line in 2D: $y - y_0 = m(x - x_0)$



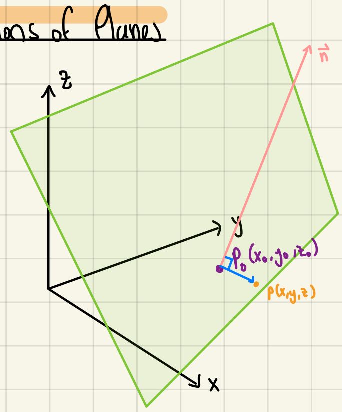
- Vector Equation of Line: $\vec{r} = \vec{r}_0 + t\vec{v}$



- [ ] In 2D, choose $\vec{v} = \langle 1, m \rangle$
  - $\vec{r} = \vec{r}_0 + t\vec{v}$ → $x = x_0 + t1$, $y = y_0 + tm$

- [ ] Given a point on the plane and a normal to the plane, write the vector equation of the plane.

Equations of Planes



- Equation of Plane: $\vec{n} \cdot \overrightarrow{P_0P} = 0$

Example: $P_0(1,2,3)$, $\vec{n} = \langle 4,5,6 \rangle$

$\overrightarrow{P_0P} = \langle x-1, y-2, z-3 \rangle$

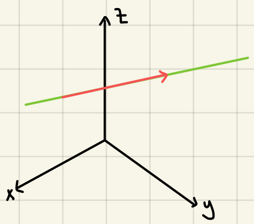
$\vec{n} \cdot \overrightarrow{P_0P} = \langle 4,5,6 \rangle \cdot \langle x-1, y-2, z-3 \rangle$

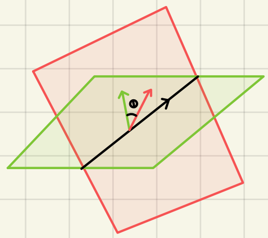
$= (4x-4) + (5y-10) + (6z-18) = 0$

$= 4x + 5y + 6z = 32$

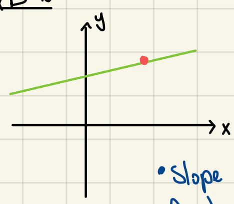
- [ ] Different forms of equations for a line in space
- [ ] Distance of a point from a line

Lines and Planes in Space





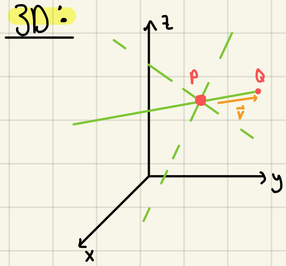
2D:



- Slope
- Point

- [ ] $y = mx + b$, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- [ ] $y - y_1 = m(x - x_1)$
- [ ] $y - y_1 = \underbrace{\dfrac{y_2 - y_1}{x_2 - x_1}}_{m}(x - x_1)$

3D:



- Point
- A vector that L is parallel to.

Suppose L is a line that passes through $P(x_0, y_0, z_0)$ and is parallel to a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in space.

Let $Q(x, y, z)$ be any other (general) point on the line.

From the figure:

$$\overrightarrow{PQ} = t\vec{v}, \; t \in \mathbb{R}.$$

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle v_1, v_2, v_3 \rangle$$

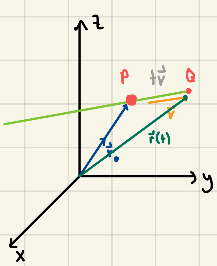
$x - x_0 = tv_1$
$y - y_0 = tv_2$
$z - z_0 = tv_3$

$\Rightarrow$

$x = x_0 + tv_1$
$y = y_0 + tv_2$
$z = z_0 + tv_3$

parametric equations for a line in space.

Vector $\vec{v}$ is called the direction vector of the line.



Notice: That t is multiplied with component of $\vec{v}$.

From here...

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

is the vector form of equation of a line.
Where $\vec{r}_0$ is the position vector of the point P and $\vec{r}(t)$ is the position vector for Q.

Example 1

Find parametric equations for the line through $(-2,0,4)$ parallel to $v = 2i + 4j - 2k$.

The equation of the line is:

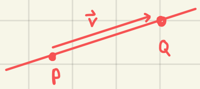
$$x = x_0 + tv_1 = -2 + 2t$$
$$y = y_0 + tv_2 = 0 + 4t$$
$$z = z_0 + tv_3 = 4 - 2t$$

Example 2

Find the parametric equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$.

The direction vector in this case can be represented by...

$$\vec{v} = \vec{PQ} = \langle 4, -3, 7 \rangle$$



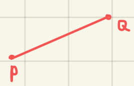
The equation of the line is...

$$x = -3 + 4t$$
$$y = 2 - 3t$$
$$z = -3 + 7t$$

Example 3

Parametrize the line segment joining the points $P(-3,2,-3)$ and $Q(1,-1,4)$.

We had...



$$x = -3 + 4t$$
$$y = 2 - 3t$$
$$z = -3 + 7t$$

$$0 \le t \le 1$$

We had...

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

($\vec{r}(t)$ → position at any time t; $\vec{r}_0$ → Initial Position; t → time; $\vec{v}$ → velocity)

$$dx/dt = v$$
$$x(t) = x_0 + vt$$

Instead,

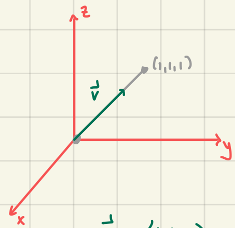
$$\vec{r}(t) = \vec{r}_0 + t|\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

(t → time; $|\vec{v}|$ → speed; $\frac{\vec{v}}{|\vec{v}|}$ → direction of motion)

Example 4

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1,1,1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 seconds.

We have... $\vec{r}(t) = \vec{r}_0 + t|\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right)$



$\vec{r}(t) = \langle 0,0,0 \rangle + t(60)\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$

$\vec{v} = \langle 1,1,1 \rangle \Rightarrow \vec{v}/|\vec{v}| = \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$

$\Rightarrow \vec{r}(t) = t\langle 60/\sqrt{3}, 60/\sqrt{3}, 60/\sqrt{3} \rangle$

$\Rightarrow \vec{r}(t) = t\langle 20\sqrt{3}, 20\sqrt{3}, 20\sqrt{3} \rangle$

Now at $t = 10$:

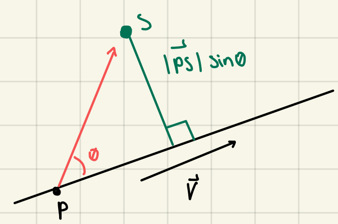
$\vec{r}(10) = 200\sqrt{3}\langle 1,1,1 \rangle$

Distance $= |\vec{r}(10)| = 600$ ft

$\vec{r}(t) = \vec{r}_0 + t|\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right)$

The Distance from a Point to a Line in Space

We want to find the distance of S from a line that passes through P and has $\vec{v}$ as the direction vector.



Required distance $= |\vec{PS}|\sin\theta$

$= \frac{|\vec{PS}|\sin\theta \cdot |\vec{v}|}{|\vec{v}|}$

Note $\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}$

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

$\Rightarrow$ distance $= \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$

Example 5

Find the distance from the point S(1,1,5) to the line

$L: x = 1+t, \quad y = 3-t, \quad z = 2t$

$x = 1+t$ → $P = (1,3,0)$ $\Rightarrow \vec{PS} = \langle 0,-2,5 \rangle$

$y = 3-t$

$z = 0+2t$ → $\vec{v} = \langle 1,-1,2 \rangle$

↓

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(1) - \hat{j}(-5) + \hat{k}(2) = \langle 1, 5, 2 \rangle$$

$$|\vec{PS} \times \vec{v}| = \sqrt{1+25+4} = \sqrt{30}, \quad |\vec{v}| = \sqrt{1+1+4} = \sqrt{6}$$

Finally, distance of S from L $= \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$

distance $= \sqrt{5}$

Summary

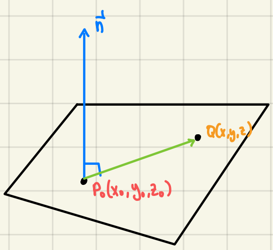
$$\left.\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}\right]$$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} \,\}; \quad \vec{r}(t) = \vec{r}_0 + t|\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right); \quad d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

- ☐ Different forms of equation for a plane in space
- ☐ Distance of a point from a plane
- ☐ Angle between two planes

Lines & Planes in Space

An Equation for a Plane is Space



Suppose we have a plane that contains the point $P_0(x_0, y_0, z_0)$.
Suppose $\vec{n}$ is the normal vector of the plane.
Let Q be a general point on the plane.
The vector $\vec{P_0Q}$ is also contained in the plane.

- A point that plane passes through.
- Normal Vector.

$\vec{P_0Q}$ and $\vec{n}$ are then orthogonal to each other.
Then...

$\vec{n} \cdot \vec{P_0Q} = 0$ (must) $\longrightarrow$ Vector form of the equation of a plane.

$$\langle a,b,c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$\Rightarrow$ $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ $\longrightarrow$ Equation of a plane written in component form.

$\langle a,b,c \rangle = \vec{n}$

$\Rightarrow$ $ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d}$ $\longrightarrow$ $ax + by + cz = d$

Example 6

Find an equation for th plane that passes through $P_0(-3,0,7)$ and is perpendicular to $n = 5i + 2j - 1k$.

The equation of th plane is:

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$\Rightarrow 5(x+3) + 2(y-0) - 1(z-7) = 0$

$\Rightarrow 5x + 2y - z = -22$

$y = mx + b$

$y - y_1 = m(x - x_1)$

using $ax + by + cz = d$

$5x + 2y - 1z = d$, take $x = -3, y = 0, z = 7$

$\Rightarrow -15 + 0 - 7 = d$

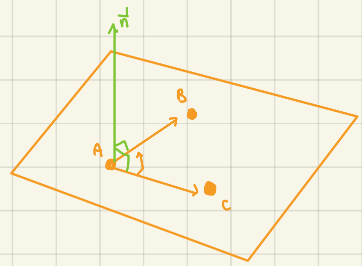
$\Rightarrow d = -22$

Example 7

Find an equation for th plane through $A(0,0,1)$, $B(2,0,0)$, and $C(0,3,0)$.

$\overrightarrow{AB} = \langle 2,0,1 \rangle$

$\overrightarrow{AC} = \langle 0,3,-1 \rangle$



The normal Vector $= \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$= \hat{i}(+3) - \hat{j}(-2) + \hat{k}(6)$

$= \langle 3,2,6 \rangle$

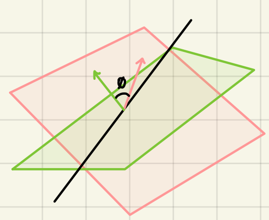
The equation of th plane is...

$3(x-0) + 2(y-0) + 6(z-1) = 0$

$\Rightarrow 3x + 2y + 6z = 6$

Example 9

Find parametric equations for th line in which th planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.



A point that the line of intersection passes through:

Take $z=0$

$3x - 6y = 15$

$2x + y = 5$ ⟶ $\Rightarrow y = 5-2x$

2nd × 2 ⟶ $12x + 6y = 30$ $\quad = 5 - 2(3)$

$15x = 45$ $\quad$ $\boxed{y=-1}$

$\boxed{x = 3}$

Therefore $(3,-1,0)$ lies on the line.

Now we need the direction vector of the line.

$\vec{n}_1 = \langle 3,-6,-2\rangle$, $\vec{n}_2 = \langle 2,1,-2\rangle$

The direction vector of the required line is...

$\vec{v} = \vec{n}_1 \times \vec{n}_2$

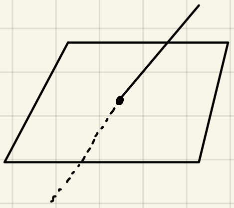
$$\vec{v} = \vec{n}_1 \times n_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \hat{\imath}(12+2) - \hat{\jmath}(-6+4) + \hat{k}(3+12)$$

$= \langle 14, 2, 15\rangle$

$$\boxed{\begin{aligned} x &= x_0 + tv_1 = 3 + 14t \\ y &= y_0 + tv_2 = -1 + 2t \\ z &= z_0 + tv_3 = 0 + 15t \end{aligned}}$$

Example 10

Find the point where the line $x = 8/3 + 2t$, $y = -2t$, $z = 1+t$ intersects the plane $3x + 2y + 6z = 6$.



Suppose the line and the plane have a point in common

Using x, y, and z from the line, substituting them into the equation of the plane.

$$3\left(8/3 + 2t\right) + 2(-2t) + 6(1+t) = 6$$

$\Rightarrow 8 + 6t - 4t + 6 + 6t = 6$

$\Rightarrow 8t = -8$

$\Rightarrow \boxed{t = -1}$

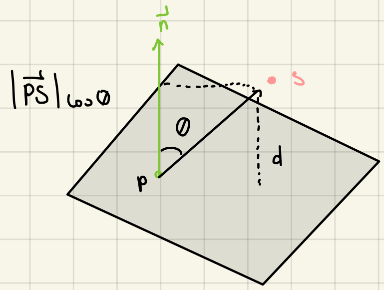
Using this in the equation of line.

↓

$x = 8/3 - 2 = 2/3$
$y = 2 \quad = 2$
$z = 1 - 1 = 0$

The point of intersection is $(2/3, 2, 0)$

Distance of a Point from a Plane



The distance of the point S from the plane that passes through P has $\vec{n}$ as the normal vector

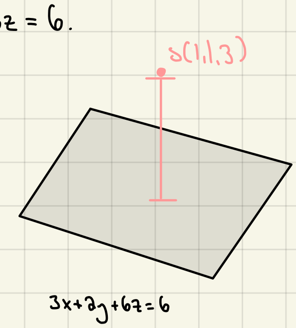
$$= d = |\vec{PS}| \cos\theta . \frac{|\vec{n}|}{|\vec{n}|}$$

$$\Rightarrow d = \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right|$$

$$= \left| \vec{PS} \cdot \frac{\vec{n}}{\vec{n}} \right|$$

Example 11

Find the distance from S(1,1,3) to the plane $3x + 2y + 6z = 6$.



$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

take $x = y = 0$
$\Rightarrow z = 1$
$P(0,0,1)$

Here $\vec{n} = \langle 3, 2, 6 \rangle$

$|\vec{n}| = \sqrt{9+4+36}$
$= 7$

$\Rightarrow \vec{PS} = \langle 1, 1, 2 \rangle$

* $P(0,3,0)$

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| \langle 1, -2, 3 \rangle \cdot \langle 3/7, 2/7, 6/7 \rangle \right|$$

$$\Rightarrow d = \left| 3/7 - 4/7 + 18/7 \right| = \boxed{\left| 17/7 \right|}$$

Angle Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normals.

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \right)$$

Example 12

Find the angle between the planes: $3x - 6y - 2z = 15$ and $2x + 1y - 2z = 5$

$\vec{n}_1 = \langle 3, -6, -2 \rangle$ $\vec{n}_2 = \langle 2, 1, -2 \rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right)$$

$\approx$ 1.38 radians or 79 degrees.

- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
- $ax + by + cz = d$

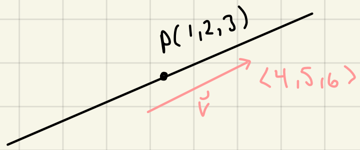
- normal vector $\vec{n} = \langle a, b, c \rangle$
- A point on the plane

$$d = \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right|$$

Quiz on 12.5

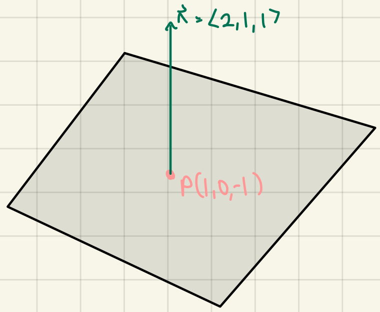
100 %

✓ 1.



$r = \langle 1,2,3\rangle + t\langle 4,5,6\rangle$ ←

✓ 2.



$ax + by + cz = d$

$2x + 1y + 1z = d$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

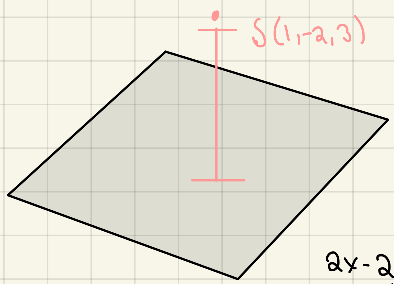
$\Rightarrow 2(x-1) + 1(y-0) + 1(z+1) = 0$

$2x - 2 + 1y + 1z + 1 = 0$

$2x + 1y + 1z = 2 - 1$

$2x + 1y + 1z = 1$ ←

✓ 3.



$2x - 2y + 1z = 3$

$P(0,0,3)$ $\quad \vec{n} = \langle 2,-2,1\rangle$

$\vec{PS} = \langle 1,-2,0\rangle$

$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$

$|\vec{n}| = \sqrt{4+4+1} = 3$

$d = \left| \langle 1,-2,0\rangle \cdot \langle 2/3, -2/3, 1/3\rangle \right|$

$= \langle 2/3, 4/3, 0\rangle$

$d = 2/3 + 4/3 + 0$

$d = 2$ ←