July 29 (Lecture 20)

Overview: We'll continue looking a bit at matrices of linear transformations and change of basis, and how diagonalization plays into these ideas. Finally, we'll end with a broad summary of the course.

Learning Goals:

- Correctly define and compute the matrix of a linear transformation from one basis to another.
- Correctly change between bases for the same vector space.
- Give a concise high-level overview of the course!

As you're getting settled:

- Homework 10 is due tomorrow (Friday, July 30), at 11:30 pm!
- I'll post some practice problems regarding the material from this past week.
- If you haven't done the CES, I'm going to give you ten minutes right now to do it. I'll hang out in a Breakout Room while you do; someone can let me know when everyone's ready and I'll come back.

  ces.uvic.ca

p. 267 **Theorem** (4.4.2)**.** *Let V be a finite-dimensional vector space with bases B and C.* Then $P_{C\leftarrow B}$ is invertible, with $P^{-1}_{C\leftarrow B} = P_{B\leftarrow C}$.

Example. Let $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ $= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.
and $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be bases for $T_{2,2}(\mathbb{R})$, the vector space of upper-triangular 2×2 matrices. Find $P_{B\leftarrow C}$ and $P_{C\leftarrow B}$, and find the B-coordinates of the matrix $\begin{bmatrix} 1 & 2 \\ \cancel{3} & \cancel{0} \end{bmatrix}$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ 0 3

Let's find $P_{C\leftarrow B} = \left[\left[\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right]_C \middle| \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right]_C \middle| \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]_C \right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \Rightarrow P_{B\leftarrow C} = P^{-1}_{C\leftarrow B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}.$$

$$\left[\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\right]_B = P_{B\leftarrow C} \left[\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\right]_C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix}.$$

Check: $4\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-3)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. ✓

$= T(p(x))$

Example. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $Tp(x) = p(-x)$, with bases $B = \{1, x, x^2\}$ and $C = \{2x-1, 3x^2+x, -2\}$.

Relative to B, T is "nice":

$$[T]_{B\leftarrow B} = \left[[T(1)]_B \,\vdots\, [T(x)]_B \,\vdots\, [T(x^2)]_B \right] = \left[[1]_B \,\vdots\, [-x]_B \,\vdots\, [x^2]_B \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \star$$

Diagonal! B is a set of eigenvectors for T. :)

$$[T]_{C\leftarrow C} = \left[[T(2x-1)]_C \,\vdots\, [T(3x^2-x)]_C \,\vdots\, [T(-2)]_C \right] = \left[[-2x-1]_C \,\vdots\, [3x^2-x]_C \,\vdots\, [-2]_C \right] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}.$$

Not nearly as nice / diagonal... But!

- $-2x-1 = -(2x-1) + (-2)$
- $3x^2 - x = 3x^2 + x + (-1)(2x-1) + \frac{1}{2}(-2)$

$$P_{B\leftarrow C} = \left[[2x-1]_B \,\vdots\, [3x^2+x]_B \,\vdots\, [-2]_B \right] = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix}.$$

$$\underbrace{\overset{D}{[T]_{B\leftarrow B}}}\,[p(x)]_B = \underbrace{\overset{P^{-1}}{P_{B\leftarrow C}}\,\overset{A}{[T]_{C\leftarrow C}}\,\overset{P}{P_{C\leftarrow B}}}\,[p(x)]_B$$

these matrices are equal! (Since $[p(x)]_B$ can be any vector in $\mathbb{R}^3$)

We changed bases to diagonalize the linear map T!

If we had diagonalized $[T]_{C\leftarrow C}$, we would have found:

So the basis is $\{3x^2 - \frac{1}{2}, -2, -4x\}$, which is okay, if not as nice as B.

$\lambda = 1$, alg. mult 2,

e-vecs $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$-\frac{1}{2}(2x-1) + 1(3x^2+x) = 3x^2 - \frac{1}{2}$

$1(-2) = -2$

$\lambda = -1$, alg. mult 1

e-vec $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \to -2(2x-1) + 1(-2) = -4x.$

$\star$ $c_1(2x-1) + c_2(3x^2+x) + c_3(-2)$

$= -2x-1,\ 3x^2-x,\ -2$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & \\ & & & \end{array}\right]$$

Example. Let $T : M_{2\cancel{\times},2}(\mathbb{R}) \to M_{2\cancel{\times},2}(\mathbb{R})$ be given by $T(A) = A^T$. Can we diagonalize T? That is, "find a basis B for $M_{2,2}(\mathbb{R})$ consisting of eigenvectors for T, so that $[T]_{B\leftarrow B}$ is diagonal".

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, to pick a simple basis. Then we have:

$$[T]_{S\leftarrow S} = \left[\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T \right]_S \middle| \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \right]_S \middle| \left[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T \right]_S \middle| \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T \right]_S \right]$$

$$= \left[\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_S \middle| \left[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right]_S \middle| \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_S \middle| \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right]_S \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ where it was easy to read the coords off because S is simple.

Let's diagonalize $[T]_{S\leftarrow S}$!

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)^2(\lambda^2-1) = (\lambda-1)^3(\lambda+1) = 0$$

$\Rightarrow \lambda = 1$ (alg. mult $=3$), -1 (alg. mult $=1$).

$\lambda = 1$: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ e-vecs for $[T]_{S\leftarrow S}$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, so for matrics:

$E_T(1) = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$\lambda = -1$: $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ e-vec is $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, giving $E_T(-1) = \text{Span}\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$.

Finally, if $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, we have $P_{S\leftarrow B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

B is a basis of e-vecs for T, and $P_{S\leftarrow B}$ diagonalizes $[T]_{S\leftarrow S}$.

$$P_{S\leftarrow B}^{-1} [T]_{S\leftarrow S} P_{S\leftarrow B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = [T]_{B\leftarrow B}.$$

Course Summary

- Started with vectors in $\mathbb{R}^n$ and their geometry! Studied the dot product, projections, norm, vector algebra, and various collections of vectors that form geometric objects like lines and planes.
- Looked at systems of linear equations as a way to solve vector equations.
- Started to look at matrices as their own objects, with matrix operations and properties; saw the connection between matrices and linear maps, and various geometric linear maps! Studied subspaces associated to matrices and linear maps, as well as invertible matrices!
- Abstracted $\mathbb{R}^n$ to general vector spaces! Linear dependence, spans, bases, dimension, and all of the other concepts from column vectors translate pretty much seamlessly to abstract vector spaces.
- Used determinants to study eigenvalues and eigenvectors of matrices (and invertible matrices!); diagonalized matrices! (See summary document of applications.)
- Saw that by using bases, we can identify linear maps with matrices, and diagonalize them too!

That's it for the class! Thanks for sticking it out; good luck on your final exam! Don't forget that you can always e-mail me or post on the forums and I'll try to get back to you soon. I plan to have office hours during exam period, so you can come chat; you can also arrange for appointments.