July 26 (Lecture 19)

Overview: First, we'll look a bit more at diagonalization and what it can do for us. Then we'll backtrack a little bit to coordinates and start seeing how we can tie together coordinates, general linear transformations, and diagonalization!

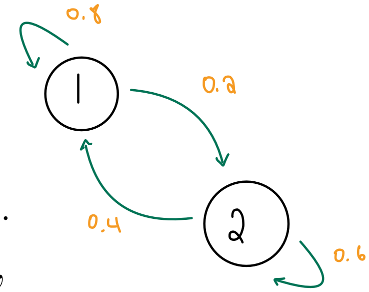
Learning Goals:

- Correctly diagonalize matrices!
- Use diagonalization to solve certain linear algebra problems.
- Correctly define the matrix of a linear transformation from one basis to another.

As you're getting settled:

- Homework 10 is due Friday, at 11:30 pm!
- Please fill out the CES! I appreciate your feedback. I'll give you 10 minutes at the start of class on Thursday to complete it, if that makes it easier for you.
- You should have your Test 2 marks back by the end of today. Please have a look at the grading and let me know before August 2nd if you have any concerns.

Example. Markov Chains (not on the final). Think: changing between two brand loyalties, or passers-by flicking a light switch on and off.



Let $P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.8 & 0.4 \\ 2 & 0.2 & 0.6 \end{array}$ "transition probability matrix". Note: column sums = 1.

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ w/ $x_i \geq 0$ and $x_1 + x_2 = 1$, then $\vec{x}$ is a "probability vector". (Theorem: $P\vec{x}$ is also a "probability vector"!)

Let's diagonalize P!

1) $\det(P - \lambda I_2) = \begin{bmatrix} 0.8-\lambda & 0.4 \\ 0.2 & 0.6-\lambda \end{bmatrix} = \lambda^2 - 1.4\lambda + 0.48 - 0.08 = (\lambda - 1)(\lambda - 0.4) = 0$

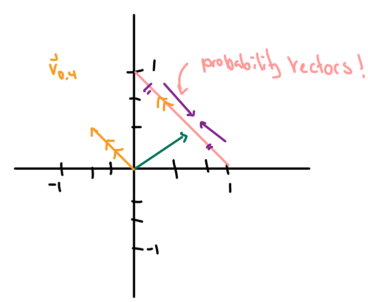
$\Rightarrow \lambda = 1, 0.4$ (P is diagonalizable)!

2) $\lambda = 1$: $\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{E}_P(1) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

$= \text{span}\left\{ \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \right\}$.

$\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ is invariant: $P\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = 1\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$. "steady state" for the Markov Chains.

$\lambda = 0.4$: $\begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{E}_P(0.4) = \text{span}\left\{ \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \right\}$.



All probability vectors look like $\vec{x} = \vec{v}_1 + c\vec{v}_{0.4}$

$\Rightarrow P\vec{x} = P\vec{v}_1 + cP\vec{v}_{0.4} = \vec{v}_1 + c(0.4)\vec{v}_{0.4}$

$\vdots$

$\Rightarrow P^k\vec{x} = \vec{v}_1 + c\underbrace{(0.4)^k}_{\to 0}\vec{v}_{0.4} \xrightarrow[k\to\infty]{} \vec{v}_1$.

Sect. 4.4, 6.6

Matrices of Linear Transformations

Let's return to general linear transformations. We saw that linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$ were determined by what they did to the standard basis. Is that true in general? Can we extend the "standard matrix" idea to general linear maps?

[No p. ref :(] **Theorem.** *Let V and W be vector spaces, let T and S be linear transformations from V to W, and let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V.* If $T(\vec v_i) = S(\vec v_i)$ for all i, then $T = S$.

Proof. $T = S$ when $T(\vec v) = S(\vec v)$ for all $\vec v \in V$. So, if $\vec v = \alpha_1 \vec v_1 + \dots + \alpha_n \vec v_n$, then we have:

$$T(\vec v) \overset{\text{by linearity}}{=} \alpha_1 T(\vec v_1) + \dots + \alpha_n T(\vec v_n)$$
$$= \alpha_1 S(\vec v_1) + \dots + \alpha_n S(\vec v_n) \underset{\text{by linearity}}{=} S(\vec v).$$

∎

Now, suppose that $L : V \to W$ is a linear map, $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V, and $C = \{\vec{w}_1, \dots, \vec{w}_m\}$ be a basis for W. What are the C-coordinates of $L(\vec v)$? (First, a helping theorem.)

p. 266 **Theorem** (4.4.1)**.** *Let V be a vector space with basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$.* Then for $\vec x, \vec y \in V$, $s, t \in \mathbb{R}$, we have $[s\vec x + t\vec y]_B = s[\vec x]_B + t[\vec x]_B$.

(ie. $[\cdot]_B$ is a linear map from V to $\mathbb{R}^n$!)

Let's try to compute $[L(\vec v)]_C$: if $[\vec v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$, then

$$[L(\vec v)]_C = [\alpha_1 L(\vec v_1) + \dots + \alpha_n L(\vec v_n)]_C \underset{(4.4.1)}{=} \alpha_1 \underline{[L(\vec v)]_C} + \dots + \alpha_n \underline{[L(\vec v_n)]_C}$$

column vectors

$$= \underbrace{\Big[\, [L(\vec v_1)]_C \;\vdots \cdots \vdots\; [L(\vec v_n)]_C \,\Big]}_{m \times n} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{[\vec v]_B}$$

p. 288 (but for us its more general!).

Definition. Let V and W be vector spaces, let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $C = \{\vec{w}_1, \dots, \vec{w}_m\}$ be bases for V and W respectively, and let $L : V \to W$ be linear. The *matrix of L from $\mathcal{B}$ to $\mathcal{C}$* is the $m \times n$ matrix $[L]_{\mathcal{C}\leftarrow\mathcal{B}}$ defined by

$$[L]_{C\leftarrow B} = \left[[L(\vec{v}_1)]_C \; \vdots \cdots \vdots \; [L(\vec{v}_n)]_C \right].$$

Example. We've already looked at $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $B = S_n$, $C = S_m$. Standard matrix: $[L]_{S_m \leftarrow S_n}$!

$$[L]_{S_m\leftarrow S_n} = \left[L(\vec{e}_1) \dots L(\vec{e}_n) \right],$$ remembering that $L(\vec{e}_i) = [L(\vec{e}_i)]_{S_m}$, since S_m is the standard basis.

$L : P_2(\mathbb{R}) \to P_3(\mathbb{R})$, $B = \{1-x, 1+x, x^2\}$, $C = \{1, x, x^2, x^3\}$.

$L(a+bx+cx^2) = (a-c) + (b+2c)x + ax^2 + (3a-b+c)x^3$. ($L$ is linear!)

Compute L of the vectors in B, then find C-coords:

$$[L(1-x)]_C = [1 - x + x^2 + 4x^3]_C = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 4 \end{bmatrix}.$$

$$[L(1+x)]_C = [1 + x + x^2 + 2x^3]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$[L(x^2)]_C = [-1 + 2x + x^3]_C = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\Rightarrow [L]_{C\leftarrow B} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}.$$

If $[p(x)]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $(p(x) = 2)$, then $[L(p(x))]_C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \end{bmatrix} \to L(p(x)) = 2 + 2x^2 + 6x^3$. ✓

From our previous computation, $[L(\vec{v})]_C = [L]_{C\leftarrow B}[\vec{v}]_B$.

What happens if we apply this definition to $L = \mathrm{id} : V \to V$? the identity map!

p. 267 **Definition.** Let V be a vector space with bases $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and C. Then the *change of coordinates* (or *change of basis*) matrix from B to C is $P_{C \leftarrow B} = [\mathrm{id}]_{C \leftarrow B} = \left[[\vec{v}_1]_C \mid \cdots \mid [\vec{v}_n]_C \right]$.

Example. $B = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}} \right\}$, $C = \left\{ \underset{\vec{u}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$.

Let's find $P_{C \leftarrow B}$: SLEs: $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}$.

Augment:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & -3 \\ 0 & 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 0 & -3 \\ & & & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -5 \\ 0 & 1 & 0 & -2 & -1 & 5 \\ 0 & 0 & 1 & 2 & 0 & -3 \\ & & & [\vec{v}_1]_C & [\vec{v}_2]_C & [\vec{v}_3]_C \end{array}\right] \implies P_{C \leftarrow B} = \begin{bmatrix} 1 & 2 & -5 \\ -2 & -1 & 5 \\ 2 & 0 & -3 \end{bmatrix}.$$

Example. $B = \{1-x, 1+x, x^2\}$, $C = \{1-x^2, x-1, 1+x+x^2\}$ bases for $P_2(\mathbb{R})$.

To compute $P_{C \leftarrow B}$: $c_1(1-x^2) + c_2(x-1) + c_3(1+x+x^2)$

$= (c_1 - c_2 + c_3) + (c_2 + c_3)x + (c_3 - c_1)x^2 = 1-x$, and $1+x$, and x^2!

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2/3 & -2/3 \\ 0 & 1 & 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & -2/3 & 1/3 \end{array}\right].$$

How do we convert from C to B, if we know how to convert from B to C? Well ... maybe $P_{C \leftarrow B}$ is invertible?