July 22 (Lecture 18)

Overview: We'll continue looking at eigenvalues and eigenvectors of matrices. Then, we'll use them to formalize what we meant when we said "this basis was *nice* for a given matrix".

Learning Goals:

- Compute eigenvalues and eigenvectors of matrices.
- Precisely define what it means to diagonalize a matrix.
- Correctly diagonalize matrices!

As you're getting settled: ◦ Test 2 marking is still ongoing (sorry).

- You'll (hopefully) have all of the material for Homework 10 after today.
- Reflection 12 will be available at the end of our class today.
- Please fill out the CES! I appreciate your feedback.
- Friday, July 23 Office Hours will be 12:00-1:00 pm instead of 11:30-12:30.

Section 6.2

Diagonalization

We know: $A\vec{v}_1 = 2\vec{v}_1$, $A\vec{v}_2 = -3\vec{v}_2$, $A\vec{v}_3 = 1\vec{v}_3$.

For $A = \begin{bmatrix} 5 & 12 & -6 \\ 0 & -3 & 0 \\ 2 & -2 & -2 \end{bmatrix}$, the basis $\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}} \right\}$ was "nice". Let's elaborate on that idea.

Let $P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$. We can compute AP:

$AP = [A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] = [2\vec{v}_1\ -3\vec{v}_2\ 1\vec{v}_3] = \begin{bmatrix} 4 & 0 & 3 \\ 0 & -3 & 0 \\ 2 & -6 & 2 \end{bmatrix}$. Since B is a basis for $\mathbb{R}^3$, P is invertible. Since $P^{-1}P = I_3$,

We have $P^{-1}\vec{v}_i = \vec{e}_i$ ($P^{-1}[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = [P^{-1}\vec{v}_1\ P^{-1}\vec{v}_2\ P^{-1}\vec{v}_3] = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$).

Thus, $P^{-1}AP = [2P^{-1}\vec{v}_1 \,\vdots\, -3P^{-1}\vec{v}_2 \,\vdots\, 1P^{-1}\vec{v}_3] = [2\vec{e}_1\ -3\vec{e}_2\ 1\vec{e}_3] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We see that $P^{-1}AP$ indicates how A "really" interacts with $\mathbb{R}^3$, when we consider the "important" directions for A $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

In terms of coordinates: if $[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, then $[A\vec{v}]_B = \begin{bmatrix} 2c_1 \\ -3c_2 \\ 1c_3 \end{bmatrix}$.

p.361 **Definition.** Let $A, B \in M_{n,n}(\mathbb{R})$. We say that A and B are *similar* when there exists an invertible matrix P such that $P^{-1}AP = B$ (or $AP = PB$).

p. 361 **Theorem** (6.2.1)**.** *Let $A, B \in M_{n,n}(\mathbb{R})$ be similar matrices. Then A and B have:*

1. The same determinent
2. The same eigenvalues
3. The same rank (and nullity)
4. The same trace (the sum of the TL-BR diagonal entries).

Example.

For $A = \begin{bmatrix} 5 & 12 & -6 \\ 0 & -3 & 0 \\ 2 & -2 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$:

det: -6, e-vals: $2, -3, 1$, rank $= 3$, trace $= 0$ ✓

[If $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, for $Q = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we have

$EQ = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$ and $QF = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$.

$\det(Q) = 2 \neq 0$, so Q is invertible, so E and F are similar!

$\Rightarrow$ det $= 0$; eigenvalues are $2, 0$; rank $= 1$; trace $= 2$.]

p. 361 **Definition.** Let $A \in M_{n,n}(\mathbb{R})$. We say that A is *diagonalizable* when A is similar to a diagonal matrix. In other words, there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Example.

A is diagonalizable! With $P = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

... How can we tell, in general?

p. 362 **Theorem** (6.2.2). *Let $A \in M_{n,n}(\mathbb{R})$.* A is diagonalizable (over $\mathbb{R}$) if and only if there is a basis for $\mathbb{R}^n$ consisting of e-vecs for A.

Proof idea. ($\Rightarrow$) If $P^{-1}AP = D$ for some $P = [\vec{v}_1, \dots, \vec{v}_n]$, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of eigenvectors!

($\Leftarrow$) If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of e-vecs (for A), then set $P = [\vec{v}_1, \dots, \vec{v}_n]$.

"$\square$"

Procedure. To diagonalize a matrix A (if possible):

1. Compute e-vals for A (using characteristic polynomial $C_A(\lambda)$)
2. For each e-val λ, compute a basis for $\mathcal{E}_A(\lambda)$.

Decision Point

3. If you don't have enough LI e-vecs (or there are complex e-vals), then A is <u>not</u> diagonalizable (over $\mathbb{R}$).
4. Otherwise, A <u>is</u> diagonalizable. Put the LI e-vecs into a matrix P as columns.
5. $D = P^{-1}AP$ is a diagonal matrix whose entries are the e-vals of A corresponding to the columns of P. (don't need to compute $P^{-1}AP$ except to check our answers).

p. 363 **Theorem.** *Let $A \in M_{n,n}(\mathbb{R})$.*

(6.2.3) A is diagonalizable if and only if for each e-val λ of A, the algebraic and geometric multiplicities of λ are equal.

(6.2.4) If A has n distinct (real) eigenvalues, then A is diagonalizable. (over $\mathbb{R}$).

HW10
Question 5

Example. Let $G = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{bmatrix}$. Determine whether or not G is diagonalizable, and if it is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}GP = D$.

1). Find e-vals! $\det(G - \lambda I_3) =$

$$\det \begin{bmatrix} -1-\lambda^{+} & 2^{-} & -2 \\ 0 & 1-\lambda^{+} & 0 \\ -2 & 2 & -1-\lambda \end{bmatrix} \overset{\text{2nd row exp!}}{=} (1-\lambda)\det\begin{bmatrix} -1-\lambda & -2 \\ -2 & -1-\lambda \end{bmatrix} = (1-\lambda)(\lambda^2+2\lambda+1-4)$$

$$= (1-\lambda)(\lambda^2+2\lambda+3) = (1-\lambda)(\lambda+3)(\lambda-1)$$

$$= -(\lambda-1)^2(\lambda+3) = 0$$

$\Rightarrow \lambda = 1$ (alg. mult $= 2$),
$\lambda = 3$ (alg mult $= 1$)

2). Find e-vecs!

$\lambda = 1$: $\begin{bmatrix} -2 & 2 & -2 \\ 0 & 0 & 0 \\ -2 & 2 & -2 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_G(1) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$ geo mult $= 2$

$\lambda = -3$: $\begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_G(-3) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$ geo mult $= 1$

3/4). By Thorem 6.2.3, alg mult = geo mult for all e-vals, so G is diagonalizable!

5). $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible, and $P^{-1}GP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = D.$

$\lambda = 1$ $\quad \lambda = -3$

Example. Let $H = \begin{bmatrix} 3 & -2 \\ 2 & 7 \end{bmatrix}$. Determine whether or not H is diagonalizable, and if it is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}HP = D$.

1) Find e-vals: $\det(H - \lambda I_2) = \det\begin{bmatrix} 3-\lambda & -2 \\ 2 & 7-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 21 + 4 = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2 = 0.$

$\Rightarrow \lambda = 5$ (alg mult $= 2$).

2) Find $\mathcal{E}_H(5)$: $\begin{bmatrix} -2 & -2 \\ 2 & 5 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{E}_H(5) = \overset{\text{geo mult}=1.}{\operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}.$

3) alg mult of $\lambda = 5$ $>$ geo mult of $\lambda = 5$, therefore it is <u>not</u> diagonalizable. ☹

Applications: Section 6.3

Example. Let $K = \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$. Show that K is diagonalizable. What is K^{10}?

1) $\det(K - \lambda I_2) = \det\begin{bmatrix} 3-\lambda & 1 \\ 5 & 7-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 25 - 5 = \lambda^2 - 10\lambda + 16 = (\lambda-8)(\lambda-2) = 0.$

$\Rightarrow \lambda = 8, 2$, both w/ alg mult $= 1$. $\Rightarrow$ K is diagonalizable by, Theorem 6.2.4.

2) $\lambda = 8$: $\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & -1/5 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{E}_K(8) = \overset{\text{geo mult } 1}{\operatorname{span}\left\{\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right\}}.$

Step 3/4

$\lambda = 2$: $\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{E}_K(2) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$

5) If $P = \begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}$, then $P^{-1}KP = D$, $\Rightarrow K^{10} = (PDP^{-1})^{10} = PD\overset{I_2}{P^{-1}P}D\overset{I_2}{P^{-1}}\overset{I_2}{\ldots}PDP^{-1}$

or $K = \underline{PDP^{-1}}$.

$D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$,

$= (PD^{10}P^{-1})$

$= P\begin{bmatrix} 8^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}P^{-1}$

★ Could used "n" instead of "10"! General!