July 19 (Lecture 17)

Overview: Our first task today is to finish up working with determinants, including seeing how they relate to invertibility. After that, we'll move on to our last big section of content: eigenvalues and eigenvectors of matrices!

Learning Goals:

- Relate determinants to invertibility.
- Precisely define and compute eigenvalues and eigenvectors of matrices.

As you're getting settled: • CES is available!

- Homework 9 is due tomorrow (Tuesday), at 11:30 pm.
- Homework 10 will be out tomorrow, due **Friday, July 30**, at 11:30 pm. It will be on material from last week and this week.
- After that, our final exam is on **Thursday, August 12**. More information to come closer to the beginning of the exam period.

Eigenvalues and Eigenvectors

Example. Let $A = \begin{bmatrix} 5 & 12 & -6 \\ 0 & -3 & 0 \\ 2 & -2 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathcal{S}_3 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$,

and $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}_3} \right\}$. $\mathcal{S}$ and $\mathcal{B}$ are both bases for $\cancel{\mathbb{C}}^3$ $\mathbb{R}$.

- Surname A-F: compute $A\vec{v}_i$. Surname G-M: compute $A\vec{e}_i$.
- Surname N-S: compute $D\vec{v}_i$. Surname T-Z: compute $D\vec{e}_i$.

$A\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = 2\vec{v}_1 \qquad A\vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix} = -3\vec{v}_2 \qquad A\vec{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = 1\vec{v}_3$

$A\vec{e}_1 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} \qquad A\vec{e}_2 = \begin{bmatrix} 12 \\ -3 \\ -2 \end{bmatrix} \qquad A\vec{e}_3 = \begin{bmatrix} -6 \\ 0 \\ -2 \end{bmatrix}$ ← no nice relationships to original vectors (either B or S).

$D\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \qquad D\vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \qquad D\vec{v}_3 = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$

$D\vec{e}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\vec{e}_1 \qquad D\vec{e}_2 = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} = -3\vec{e}_2 \qquad D\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\vec{e}_3$

Take-away. The basis is "nice" for A, and the basis S is "nice" for D.

$\lambda \in \mathbb{R}$ [Focusing on real quantities]

p. 348 **Definition.** Let $A \in M_{n,n}(\mathbb{R})$. If $\vec{v} \in \mathbb{R}^n$ is a non-zero vector such that $A\vec{v} = \lambda\vec{v}$, then we say that λ is an eigenvalue (e-val) for A and $\vec{v}$ is an eigenvector (e-vec) for A corresponding to λ. (Sometimes we say that $(\lambda, \vec{v})$ is an eigenvalue-eigenvector pair).

Example. For A, $2, -3$, and 1 are e-vals, w/ e-vecs $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

For D, also $2, -3, 1$ are e-vals, w/ e-vecs $\vec{e}_1, \vec{e}_2, \vec{e}_3$ respectively.

For I_n: Let $\vec{v} \in \mathbb{R}^n$ be non-zero. Then $I_n\vec{v} = \vec{v} = 1 \cdot \vec{v}$. So, 1 is an e-val for I_n, and every non-zero vector is an e-vec for I_n, corresponding to 1.

Example. Do eigenvalue/eigenvectors always exist, for any matrix?

Let $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \vec{0}$. Then:

$B\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, and $B\vec{v} \cdot \vec{v} = (-b)(a) + (a)(b) = 0$. Thus, $B\vec{v}$ and $\vec{v}$ are orthogonal!

$B\vec{v} \neq \vec{0}$ (since $\vec{v} \neq \vec{0}$), and so $B\vec{v}$ is not a scalar multiple of $\vec{v}$.

... so B has no real e-vals/e-vecs! ‼

But: $B\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}$, So B has complex e-vals/e-vecs, $\lambda = \pm i$ w/ $\vec{v} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$.

$i^2 = -1$ $B\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = (-i)\begin{bmatrix} -i \\ 1 \end{bmatrix}$,

Take-away: We'll stick to real e-vals/e-vecs in this course, but the complex numbers are the "natural habitat" for e-vals/e-vecs.

Finding eigenvalues/eigenvectors: For example, let $X = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$.

To find ↗, we need to solve $X\vec{v} = \lambda\vec{v}$ for $\underline{\lambda}$ and $\underline{\vec{v}}$.

<u>Rewrite</u>: $X\vec{v} - \lambda\vec{v} = \underline{X}\vec{v} - \underline{\lambda I_2}\vec{v} = \underbrace{(X - \lambda I_2)}_{\text{"}Y\text{"}}\vec{v} = \vec{0}$. $\longrightarrow Y\vec{v} = \vec{0}$.

To guarantee that there is a non-zero solution $\vec{v}$ to ↑, then we require $\text{Null}(X - \lambda I_2) \neq \{\vec{0}\}$ (i.e. is non-trivial).

This condition is equivalent, via Theorem 3.5.4, to $X - \lambda I_2$ being <u>non</u>-invertible, <u>or</u> ... $\det(X - \lambda I_2) = 0$.
This new condition depends <u>only</u> on λ, so let's solve for λ!

$$0 = \det(X - \lambda I_2) = \det\begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) - 4 = -2 - 2\lambda + \lambda + \lambda^2 - 4 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

$\Rightarrow \lambda = 3, -2$ are the <u>only</u> e-vals for X!

Sub λ back in to solve for $\vec{v}$:

$\lambda = 3$: $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an e-vec corresponding to 3.

$\lambda = -2$: $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \vec{v}_{-2} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$, <u>or</u> $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, are both e-vecs corresponding to -2.

p.350 **Theorem** (6.1.1). *Let $A \in M_{n,n}(\mathbb{R})$.* A [real] number λ is an e-val for A if and only if $\det(A - \lambda I_n) = 0$.

If λ is an eigenvalue of A, then all non-zero solutions to $(A - \lambda I_n)\vec{v} = \vec{0}$ are all of the eigenvectors for A corresponding to λ.

Example. For $A = \begin{bmatrix} 5 & 12 & -6 \\ 0 & -3 & 0 \\ 2 & -2 & -2 \end{bmatrix}$, we have: $\det(A - \lambda I_3) = \det \begin{bmatrix} 5-\lambda & 12 & -6 \\ 0 & -3-\lambda & 0 \\ 2 & -2 & -2-\lambda \end{bmatrix}$

$$= -(3+\lambda)\det\begin{bmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{bmatrix} = -(\lambda+3)\big((5-\lambda)(-2-\lambda)+12\big)$$

$$= -(\lambda+3)(-10+2\lambda-5\lambda+\lambda^2+12) = -(\lambda+3)(\lambda^2-3\lambda+2)$$

$$= -(\lambda+3)(\lambda-2)(\lambda-1) = 0.$$

$\Rightarrow$ e-vals of A are $-3, 2, 1$ (as we saw before)!

Computation of eigenvectors for A

$A = \begin{bmatrix} 5 & 12 & -6 \\ 0 & -3 & 0 \\ 2 & -2 & -2 \end{bmatrix}$, $\lambda = -3, 2, 1$. What are the eigenvectors?

$\lambda = -3$: $A-(-3)I_3 = \begin{bmatrix} 8 & 12 & -6 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$. $\Rightarrow \mathrm{Null}(A+3I_3) = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}\right\}$, so can choose $\vec{v}_{-3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$. ✓

$\lambda = 2$: $A-(2)I_3 = \begin{bmatrix} 3 & 12 & -6 \\ 0 & -5 & 0 \\ 2 & -2 & -4 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\Rightarrow \mathrm{Null}(A-2I_3) = \mathrm{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$, so choose $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. ✓

$\lambda = 1$: $A-(1)I_3 = \begin{bmatrix} 4 & 12 & -6 \\ 0 & -4 & 0 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\Rightarrow \mathrm{Null}(A-I_3) = \mathrm{span}\left\{\begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}\right\}$, so choose $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$. ✓

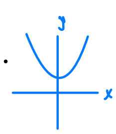
p.350 **Definition.** Let λ be an eigenvalue for A. The *eigenspace* for A corresponding to λ, denoted E_λ is the set of all eigenvectors for A corresponding to λ, as well as the zero vector.

In short: $E_\lambda = E_A(\lambda) = \mathrm{Null}(A - \lambda I)$.

p.352 **Definition.** Let $A \in M_{n,n}(\mathbb{R})$. The *characteristic polynomial* of A is $\det(A - \lambda I)$.

Notation: · Textbook: $C(\lambda)$
· Joseph: $C_A(\lambda)$ (to specify the matrix).

Example. Let $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ be as before.



$C_B(\lambda) = \det(B - \lambda I) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$, which has no real roots

$\Rightarrow$ No real e-vals!

[But: B has complex e-vals, as $\lambda^2+1 = (\lambda - i)(\lambda + i)$.

$\lambda = i$: $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$, so $E_B(i) = \mathrm{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$. Similarly, $E_B(-i) = \mathrm{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$.]

Example. Let $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. What is $C_C(\lambda)$?

$C_C(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix} = (-\lambda)^3 = -\lambda^3$. Setting $C_C(\lambda) = 0$ says that the only e-val for C is $\lambda = 0$. (Repeated 3 times!)

What is $\mathcal{E}_C(0)$?

$C - 0 I_3 = C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, in RREF! $\Rightarrow \mathcal{E}_C(0) = \text{Null}(C - 0I_3) = \text{Null}(C)$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. dimension = 1.

p. 354 **Definition.** Let $A \in M_{n,n}(\mathbb{R})$ have an eigenvalue λ. The *algebraic multiplicity* of λ is the number of times λ is repeated as a root of $C_A(\lambda)$.

$[\alpha_A(\lambda)]$

The *geometric multiplicity* of λ is the dimension of the eigenspace for λ: $\dim(\mathcal{E}_A(\lambda))$. $[\gamma_A(\lambda)]$

Example.

- For A, alg. mult. of $\lambda = 2, -3, 1$ were all $= 1$.
  geo. mult. of $\lambda = 2, -3, 1$ were all $= 1$.

- For C, alg. mult of $\lambda = 0$ was 3.
  geo. mult of $\lambda = 0$ was only 1.

[• For B, alg. mult of $\lambda = i, -i$ are both 1.
geo. mult of $\lambda = i, -i$ are both 1.]

Note: What's the degree of $C_A(\lambda)$? Turns out, it's equal to n!
So the sum of the algebraic multiplicities is also equal to n.

p. 355 **Theorem.** *Let $A \in M_{n,n}(\mathbb{R})$.*

(6.1.2) If λ is an e-val for A, then $1 \leq$ geo. mult. of $\lambda \leq$ alg. mult. of λ.

(6.1.3) If $\lambda_1, \ldots, \lambda_k$ are distinct e-vals for A, with e-vecs $\vec{v}_1, \ldots, \vec{v}_k$ corresponding to $\lambda_1, \ldots, \lambda_k$ respectively, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

p. 355 **Theorem** (6.1.4 (3.5.4))**.** *Let $A \in M_{n,n}(\mathbb{R})$. The following are equivalent:*

1. A is invertible.

11. 0 is <u>not</u> an eigenvalue for A.

Proof. A is invertible $\iff \text{Null}(A) = \{\vec{0}\} = \text{Null}(A - 0I_n) = \mathcal{E}_A(0)$
$\iff$ 0 is <u>not</u> an e-val for A.

□

Example.

- By Theorem 6.1.4, since we know that A, B, X did not have 0 as an eigenvalue, we see that A, B, X are all invertible.

On the other hand, C is <u>not</u> invertible (which we could see using RREF, etc).

- For A, we could compute the eigenvectors $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}$, and by Theorem 6.1.3, this set of vectors is LI, so we obtain "for free" (Basis Thm) that it's actually a basis for $\mathbb{R}^3$.

Eigenvalues/Eigenvectors of Linear Maps: We can define e-vals and e-vecs for linear maps in exactly the same way as for matrices (hence the copy/paste of the definition)!

Definition. Let $L : V \to W$ be a linear map. If $\vec{v} \in V$ is a non-zero vector such that $L(\vec{v}) = \lambda\vec{v}$, then we say that λ is an *eigenvalue* (or e-val) for L and $\vec{v}$ is an *eigenvector* (or e-vec) for L corresponding to the eigenvalue λ. The pair $(\lambda, \vec{v})$ is sometimes called an eigenvalue-eigenvector pair.

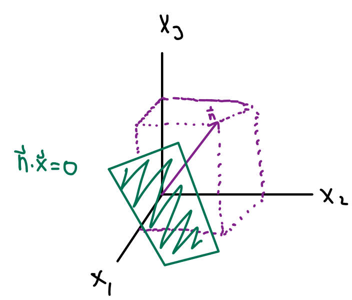
Example. Consider the projection map $\text{proj}_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ where $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note that $\text{proj}_{\vec{n}}(\vec{n}) = \frac{\vec{n}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \vec{n} = 1\vec{n}$, so 1 is an eigenvalue for $\text{proj}_{\vec{n}}$! With e-vec $\vec{n}$.

Then, if $\vec{v}$ is orthogonal to $\vec{n}$, we have $\text{proj}_{\vec{n}}(\vec{v}) = \frac{\vec{n}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \vec{0} = 0\vec{n}$. So, all non-zero vectors orthogonal to $\vec{n}$ are <u>also</u> eigenvectors for $\text{proj}_{\vec{n}}$, corresponding to eigenvalue 0!

The plane $\vec{n}\cdot\vec{x} = 0$ is the eigenspace $E_{\text{proj}_{\vec{n}}}(0)$.

★Writing $\vec{v} = \text{proj}_{\vec{n}}(\vec{v}) + \text{perp}_{\vec{n}}(\vec{v})$ decomposes $\vec{v}$ into a sum of eigenvectors!



Example. Let $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the map $L_A(\vec{v}) = A\vec{v}$, with $A = \begin{bmatrix} 5 & 12 & -6 \\ 0 & -3 & 0 \\ 2 & -2 & -2 \end{bmatrix}$.

What do the eigenvectors look like?