July 12 (Lecture 16)

Overview: After finishing up with linear maps between abstract vector spaces, we'll move on to briefly studying a tool that we'll use in the next (and last) big section of the course, the determinant of a matrix.

Learning Goals:

- Compute the range and null space of linear maps between abstract vector spaces.
- Compute determinants of square matrices in multiple ways.
- Relate determinants to invertibility.

As you're getting settled: ◦ HW8 Solutions Q1 to be fixed.

- Homework 9 is due *next* Tuesday (the 20th), at 11:30 pm.
- Test 2 is on Thursday! July 15. It will be available on Crowdmark 10:00 am to 8:00 pm Pacific time but you'll only get 1.5 hours to do it, same as last time. I've posted practice problems on Brightspace.
- You will be able to use (and are expected) to use a computer to compute RREFs and matrix-multiplication on Test 2, unless a question explicitly states otherwise.

1×1: $A = [a]$ is invertible $\iff$ $a \neq 0$, $A^{-1} = [1/a]$.

Chapter 5

Determinants

We saw that for 1×1 and 2×2 matrices...

2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff$ $ad-bc \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Can we generalize to $n\times n$ matrices?

$(m \times n)$

p 311 **Definition.** Let A be an $n \times n$ matrix. For each row i and column j, the (i, j)-th *submatrix* of A, denoted $A(i, j)$, is the $(n-1)\times(n-1)$ matrix obtained by removing row i and column j from A.

Example. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 9 \\ -1 & 5 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 4 & 3 \\ 5 & 2 & 13 \end{bmatrix}$. Compute $A(1, 3)$ and $B(2, 2)$.

$A(1,3) = \begin{bmatrix} 0 & 1 & 9 \\ -1 & 5 & 4 \end{bmatrix}$, $B(2,2) = \begin{bmatrix} 0 & 1 \\ 5 & 13 \end{bmatrix}$.

p.311 **Definition.** Let A be a square matrix ($n \times n$ for $n \geq 1$). We define the *determinant* of A, denoted $\det(A)$, recursively:

- If $n=1$, then $\det(A) = a_{11}$. ← Base Case.
- If $n\geq 2$, then $\det(A) = a_{11}C_{11} + \dots + a_{1n}C_{1n}$, where C_{ij} is the (ij)-cofactor of A, defined by $C_{ij} = (-1)^{i+j}\det(A(i,j))$.

Sign Pattern: $\begin{bmatrix} + & - & + & \\ - & + & - & \cdots \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$

☆ *Notation.* Sometimes we write $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. $\longleftarrow$ try not to do this! As it can be confusing. !!

Example. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 4 & 3 \\ 5 & 2 & 13 \end{bmatrix}$.

Compute the determinants of A, B, and C.

$\det(A) = 2((-1)^{1+1}\det[2]) + 1((-1)^{1+2}\det[1]) = 2\cdot 2 - 1\cdot 1 = 4-1 = 3.$

$\boxed{\underset{a}{} \quad \underset{d}{} \quad \underset{b}{} \quad \underset{(-c)}{} \quad = ad - bc}$

$\det(B) = 3\det[3] - 9\det[1] = 3^2 - 9 = 0.$

$\det(C) = \cancel{0\det\begin{bmatrix} 4 & 3 \\ 2 & 13 \end{bmatrix}} - 1\det\begin{bmatrix} -1 & 3 \\ 5 & 13 \end{bmatrix} + 1\det\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}$

$= -(-13-15) + (-2-20) = 28 - 22 = 6.$

Example. Let $F = \begin{bmatrix} k & 1 & 0 \\ 0 & 3 & k \\ 1 & k & -2 \end{bmatrix}$. Compute $\det(F)$. k is some real number.

$\det(F) = k\det\begin{bmatrix} 3 & k \\ k & -2 \end{bmatrix} - 1\det\begin{bmatrix} 0 & k \\ 1 & -2 \end{bmatrix} + 0\det[\cdots]$

$= k(-6-k^2) - (-k) = -5k - k^3 = -k(k^2+5).$

p. 312 **Theorem** (5.1.1)**.** The determinant of an $n \times n$ matrix may be computed by a <u>cofactor expansion</u> along <u>any</u> row or column, not just along the first row (i.e. the definition).

Row 1: two 0's, not bad.
but Col 3 has <u>three</u> 0's! ✓

Example. Let $\overset{\star}{A} = \begin{array}{c|cccc|c} & + & - & + & - & \\ \hline & 0 & 2 & 0 & 1 & + \\ & 3 & -2 & 1 & 3 & - \\ & 2 & 1 & 0 & 2 & + \\ & 2 & 1 & 0 & 5 & - \end{array}$. Compute $\det(A)$.

<u>Expand along col 3!</u>

$\det(A) = 0\,\cancel{\det}[\cdots] - \det\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} + 0\,\cancel{\det}[\cdots] - 0\,\cancel{\det}[\cdots]$

$= -\left(0\,\cancel{\det}[\cdots] - 2\det\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} + 1\,\underbrace{\det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}}_{2-2=0}\right)$

$= 2(10-4) = 2(6) = 12.$

• expand along row 1

p. 314 **Theorem.** *Let A be a square matrix.*

(5.1.2) If a row or column of A is all 0's, the $\det(A) = 0$.

(5.1.3) If A is upper or lower triangle, the $\det(A) = a_{11}a_{22}\cdots a_{nn}$
(then product of the main diagonal entries).

(5.1.4) $\det(A^T) = \det(A)$

Proof of 5.1.3. If A is upper-Δ, use 1^{st} column expansion.

$\det\begin{bmatrix} a_{11} & \rightarrow & \\ \downarrow & \ddots & \\ 0 & & a_{nn} \end{bmatrix} = a_{11}\det\begin{bmatrix} a_{22} & \rightarrow & \\ \downarrow & \ddots & \\ 0 & & a_{nn} \end{bmatrix} + 0\text{'s} \ldots = \ldots = a_{11}a_{22}\cdots a_{nn}.$

" □ "

Example.

$\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix} = 0.$

$\det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} = (1)(3)(6) = 18.$

$\det\begin{bmatrix} 0 & 3 & 2 & 2 \\ 2 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 2 & 5 \end{bmatrix}$ A^T ☆

$= \det(A^T) = 12.$

Section 5.2

More Properties of Determinants

We can use row operations to compute determinants!

p. 317, 319, 320 **Theorem.** *Let A be a square matrix.*

(5.2.1) If $A \xrightarrow{aR_i} B$, then $\det(B) = a\det(A)$. $\longrightarrow$ $\det(A) = \frac{1}{a}\det(B)$

(5.2.2) If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(B) = -\det(A)$.

(5.2.3) If A has two equal rows, then $\det(A) = 0$.

(5.2.4) If $A \xrightarrow{R_i + aR_j} B$, then $\det(B) = \det(A)$.

All of the above hold if we replace "row" by "column" (including "column operations").

Example. Let $A = \begin{bmatrix} 0 & 3 & -6 & 9 \\ 1 & -2 & 4 & -4 \\ -1 & 3 & -4 & 4 \\ 2 & \cancel{-}0 & 0 & 2 \end{bmatrix}$. Compute $\det(A)$.

$3 \det(\text{new}) = \frac{1}{3}\det(A)$

$$\det(A) \overset{\substack{\frac{1}{3}R_1}}{\underset{\substack{R_3+R_2 \\ R_4-2R_2}}{=}} 3\det\begin{bmatrix} 0^{+} & 1 & -2 & 3 \\ 1^{-} & -2 & 4 & -4 \\ 0^{+} & 1^{-} & 0^{+} & 0^{-} \\ 0 & 4 & -8 & 10 \end{bmatrix} \overset{\substack{\text{3rd} \\ \text{row} \\ \text{expansion}}}{=} (3)(-1)\det\begin{bmatrix} 0 & -2 & 3 \\ 1 & 4 & -4 \\ 0 & -8 & 10 \end{bmatrix}$$

$$\overset{\substack{\text{1st column} \\ \text{expansion}}}{=} (-3)(-1)\det\begin{bmatrix} -2 & 3 \\ -8 & 10 \end{bmatrix} \overset{R_2 - 4R_1}{=} 3\det\begin{bmatrix} -2 & 3 \\ 0 & -2 \end{bmatrix}$$

$$= (3)(-2)(-2) = 12.$$

(df 3.5.4)

p. 324 **Theorem** (5.2.5)**.** *Let A be an $n \times n$ matrix. The following are equivalent:*

2. $\mathrm{rank}(A) = n$.

10. $\det(A) \neq 0$.

Proof. ($\Rightarrow$) If $\mathrm{rank}(A)=n$, then $RREF(A) = I_n$, so $\det(A) = (\text{non-zero factors}) \cdot \underbrace{\det(I_n)}_{=1} \neq 0$.

($\Leftarrow$) If $\mathrm{rank}(A) < n$, then $RREF(A)$ has a 0 on the main diagonal, so $\det(A) = (\text{factors}) \cdot (\ldots 0) = 0$.

Example. Consider $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 4 & -1 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Which of A, B are invertible? Just compute the determinants!

$\det(A) = 4 \det \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = 4(1+6) = 28 \neq 0$. $\longrightarrow$ A is invertible!

$\det(B) = (1)(6) - (2)(3) = 6 - 6 = 0$. $\Rightarrow$ B is not invertible!

(note that $\mathrm{rank}(B) = 1 < 2$.)

Note: Unfortunately, the determinant does not tell us what A^{-1} actually is, only that it exists (or not).

p. 325 **Theorem** (5.2.7). Let $A, B \in M_{n,n}(\mathbb{R})$. Then $\det(AB) = \det(A)\det(B)$.

Example. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 & 2 \\ 4 & 2 & 0 \\ -2 & 1 & 3 \end{bmatrix}$. Compute $\det(AB)$.

Use Theorem 5.2.7!

$$\det(AB) = \det(A)\det(B) = ((1)(5)(-6)) \det \begin{bmatrix} 7 & 0 & 2 \\ 4 & 2 & 0 \\ -2 & 1 & 3 \end{bmatrix} \overset{C_1 - 2C_2}{=} -30 \det \begin{bmatrix} 7 & 0 & 2 \\ 0 & 2 & 0 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\overset{\text{2nd row op}}{=} -30(2) \det \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = -60(21+8) = -1740.$$

Example. Idea behind Theorem 5.2.7:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$. Then $RA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & -6 & -12 \end{bmatrix}$, which is the same matrix we'd get if we did $R_3 - 7R_1$.

Then $\det(RA) = \det(A)$, by Theorem 5.2.5.

All row operations can be represented by multiplication by "elementary" matrices, and if a matrix is invertible then it is actually the product of elementary matrices.

cf. Section 3.6.

So we would just repeatedly apply Theorem 5.2.1, 5.2.2, and 5.2.4 to get Theorem 5.2.7.