July 8 (Lecture 15)

Overview: We'll start today with dimension (finally). After seeing that definition and some examples, we'll extend the concept of linear maps to linear maps between abstract vector spaces.

Learning Goals:

- Precisely define and calculate dimension of a vector space.
- Explain (using theorems) how dimension relates to spanning sets, linearly independent sets, and bases.
- Define and work with linear maps between abstract vector spaces.

As you're getting settled: ◦ Reflection 10 available after class!

- Homework 8 is due tomorrow (Friday, July 9), at 11:30 pm.
- Homework 9 to come out tomorrow, due Tuesday July 20th at 11:30 pm.
- Test 2 is next Thursday! July 15. It will be available on Crowdmark 10:00 am to 8:00 pm Pacific time but you'll only get 1.5 hours to do it, same as last time. I'll try to post some targetted practice problems that I think might be helpful.
- Watch for HW or Test instructions that say "You may use a computer or calculator to perform [specific computations]"!

(not $\{\vec{0}\}$)

p. 252, 257 **Theorem.** *Let V be a non-trivial vector space with finite dimension.*

(4.3.2) If T is a spanning set of k vectors for V, then some subset of T is a basis for V.

(4.3.5) If S is a LI set in V w/ fewer than $\dim(V)$ vectors, then S can be extended to a basis for V.

(4.3.6) If W is a subspace of V, then $\dim(W) \leq \dim(V)$.

"Goldilocks Theorem"

p. 259 **Theorem** (4.3.7)**.** *Let V be a vector space with* $\dim(V) = n \geq 1$.

1. If $S \subseteq V$ has more than n vectors, the S is linearly dependent.

2. If $S \subseteq V$ has fewer than n vectors, the S cannot span V.

3. If $B \subseteq V$ has exactly n vectors then B is linearly independent if and only if B spans V.

(+4).

Example. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

Let $W = \text{span}\left\{A, A^T, B, B^T, A + B\right\}$. Find $\dim(W)$.

Solution. Observe: by part 1 of 4.3.7, since $5 > \dim(M_{2,2}(\mathbb{R})) = 4$, thus spanning set is not LI.

Note: $A+B$ is in span$\{A, B\}$, $B^T = -B$, and $A - A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{-1}{2} B$, so $A^T = A + \frac{1}{2}B \in \text{span}\{A, B\}$.

$\Rightarrow W = \text{span}\{A, B\}$.

We can show that $\{A, B\}$ is LI, so it is also a basis for W!

$\Rightarrow \dim(W) = 2$.

- If we wanted to extend $\{A, B\}$ into a basis for $M_{2,2}(\mathbb{R})$, then by 4.3.7(3), we simply need to add two more vectors to get a LI set.

(try $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$)

Section 4.5

General Linear Maps

p. 273 **Definition.** Let V and W be vector spaces. A map $L : V \to W$ is a *linear* map (or transformation) when for all $\vec{x}, \vec{y} \in V$, $s, t \in \mathbb{R}$,

we have $L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$.

Example. ∘ linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$!

∘ $L : M_{2,2}(\mathbb{R}) \to P_3(\mathbb{R})$, $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + bx + cx^2 + dx^3$. (cf. HW8, Q5(b)!).

Linear? ..

∘ $id : V \to V$, $id(\vec{x}) = \vec{x}$ (the <u>identity map</u>) is linear.

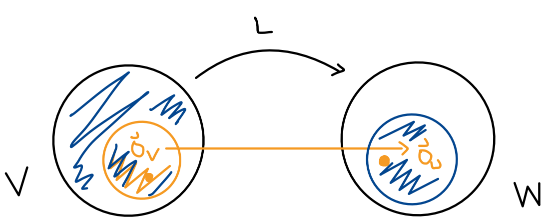
∘ $Z : V \to V$, $Z(\vec{x}) = \vec{0}_V$ (the <u>zero map</u>) is linear.

∘ <u>transpose</u> and matrix multiplication (by a fixed matrix) are both linear! (Theorem 3.1.2, 3.1.4)

∘ $\mathcal{E} : P_5(\mathbb{R}) \to P_5(\mathbb{R})$, $\mathcal{E}(a_0 + \ldots + a_5x^5) = a_0 + a_2x^2 + a_4x^4$. (only even term!)

<u>Linear?</u> $\mathcal{E}(sp(x) + tq(x)) = \mathcal{E}((sa_0 + tb_0) + \ldots + (sa_5 + tb_5)x^5)$

$$= sa_0 + tb_0 + (sa_2 + tb_2)x^2 + (sa_4 + tb_4)x^4$$
$$= s(a_0 + a_2x^2 + a_4x^4) + t(b_0 + b_2x^2 + b_4x^4)$$
$$= s\mathcal{E}(p(x)) + t\mathcal{E}(q(x)) \checkmark$$



p. 274 **Definition.** Let $L : V \to W$ be a linear map between vector spaces V and W.

- The *range* of L, denoted Range(L), is $\{L(\vec{x}) : \vec{x} \in V\} \subseteq W$. ⋆
- The *null space* of L, denoted Null(L), is $\{\vec{x} \in V : L(\vec{x}) = \vec{0}_W\} \subseteq V$ ⋆

Example. $\mathcal{E} : P_5(\mathbb{R}) \longrightarrow P_5(\mathbb{R})$, $\mathcal{E}(a_0 + \dots + a_5x^5) = a_0 + a_2x^2 + a_4x^4$.

When is $\mathcal{E}(p(x)) = 0$? $\Rightarrow a_0 = a_2 = a_4 = 0$. No restrictions on a_1, a_3, a_5.

$a_0 + a_2x^2 + a_4x^4$

$\Rightarrow \text{Null}(\mathcal{E}) = \{a_1x + a_3x^3 + a_5x^5 : a_1, a_3, a_5 \in \mathbb{R}\} = \text{span}\{x, x^3, x^5\}$.

For Range$(\mathcal{E})$, let $b_0 + b_2x^2 + b_4x^4 \in P_5(\mathbb{R})$. Every $p(x) \leftarrow P_5(\mathbb{R})$ such that $a_0 = b_0$, $a_2 = b_2$, $a_4 = b_4$ satisfies $\mathcal{E}(p(x)) = b_0 + b_2x^2 + b_4x^4$.

$\Rightarrow \text{Range}(\mathcal{E}) = \text{span}\{1, x^2, x^4\}$.

Define the trace of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $tr(A) = a + b$ (can generalize to $n \times n$).

The trace is a linear map from $M_{2\times 2}(\mathbb{R})$ to $\mathbb{R}^{(1)}$.

Null(tr): if $tr(A) = 0$, then $a + d = 0$, and no other restrictions.

SLE!

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & d \end{bmatrix} \Rightarrow \text{Null}(tr) = \text{span}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

Range(tr): What are the output values of tr? Note that $tr\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1$, and tr is linear, so for any $c \in \mathbb{R}$,

$tr\left(c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = c\,tr\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = c \cdot 1 = c$.

$\Rightarrow \text{Range}(tr) = \mathbb{R}$.

p. 276 **Definition.** Let $L : V \to W$ be a linear map between vector spaces V and W.

- The *rank* of L, denoted rank(L), is $\dim(\text{Range}(L))$.
- the *nullity* of L, denoted nullity(L), is $\dim(\text{Null}(L))$.

For $L : \mathbb{R}^n \to \mathbb{R}^m$, $\text{rank}(L) = \dim(\text{Range}(L)) = \dim(\text{col}([L]))$
$= \text{rank}([L])$.

(cf. Theorem 3.4.9).

p. 277 **Theorem** (4.5.2, Rank-Nullity). Let $L: V \to W$ be a linear map between abstract vector spaces w/ $\dim(V) = n$.
Then, $\operatorname{rank}(L) + \operatorname{nullity}(L) = n = \dim(V)$.

Proof.
First, we can find a basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\operatorname{Null}(L)$ (so $\operatorname{nullity}(L) = k$), which we can extend to a basis $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$ for V. (Theorem 4.3.5). *

Now, if $\vec{w} \in \operatorname{Range}(L)$, then by definition we have

$$\vec{w} = L(\vec{v}) = L(\cancel{a_1\vec{v}_1} + \cancel{\dots} + \cancel{a_k\vec{v}_k} + a_{k+1}\vec{w}_{k+1} + \dots + a_n\vec{w}_n)$$
$$= a_{k+1}L(\vec{w}_{k+1}) + \dots + a_nL(\vec{w}_n), \text{ since } L \text{ is linear and the } \vec{v}_i \in \operatorname{Null}(L).$$

$\Rightarrow \operatorname{Range}(L) = \operatorname{span}\{\underbrace{L(\vec{w}_{k+1}), \dots, L(\vec{w}_n)}\}$ call this set S. S spans Range(L) is S LI?

Well, if $\vec{0} = a_{k+1}L(\vec{w}_{k+1}) + \dots + a_nL(\vec{w}_n) = L(a_{k+1}\vec{w}_{k+1} + \dots + a_n\vec{w}_n)$,
then $a_k\vec{w}_{k+1} + \dots + a_n\vec{w}_n = b_1\vec{v}_1 + \dots + b_k\vec{v}_k \in \operatorname{Null}(L)$, and all of the coefficients are 0 (by*).

$\Rightarrow$ S is LI, thus a basis for Range(L), and thus $\operatorname{rank}(L) + \operatorname{nullity}(L) = n - k + k = n$.

∎

Example. $L: M_{2,2}(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$, $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a - 2b + cx^2 + (a-d)x^3$. (L is linear!)

To compute Null(L), we have: $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \Rightarrow \begin{cases} a - 2b = 0 \\ c = 0 \\ a - d = 0 \end{cases}$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & \underbrace{0}_{\text{free variable!}} \end{bmatrix}. \longrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} d \\ \frac{1}{2}d \\ 0 \\ d \end{bmatrix} = d\begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Translate back to matrices: $\operatorname{Null}(L) = \operatorname{span}\left\{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\right\}$. $\Rightarrow \operatorname{Nullity}(L) = 1$.
basis w/ 1 element.

By Rank-Nullity Theorem: $\operatorname{rank}(L) = \dim(M_{2,2}(\mathbb{R})) - 1 = 4 - 1 = 3$.

Thus, to find a basis for Range(L), we simply need to find 3 LI vectors in Range(L)!

(Try to see why $\{1, x^2, x^3\}$ works.) (Theorem 4.3.7 Part 3/4).