July 5 (Lecture 14)

Overview: Many of the same concepts we had for $\mathbb{R}^n$ hold also for abstract vector spaces! And we'll finally start to define *dimension*, and see why holding off on the definition means we can see it as a much more general concept than previously.

Learning Goals:

- Identify abstract vector spaces (and subspaces thereof) and explain how the definitions apply in different contexts.
- Work with spanning sets, linear independence, and bases in abstract vector spaces.
- Precisely define dimension of a vector space.

As you're getting settled:

- Homework 8 is out, due Friday, July 9.
- Heads up: Test 2 is next week! Thursday, July 15. We'll be using the same test format as Test 1 in terms of availability period, duration of the test, rough number of questions, etc. Material to be assessed will be finalized on Thursday.
- Great work on the reflection! Thanks for giving it a shot.
- Picking up the pace, due to our lost class. I'll be sure to post filled-in examples that get skipped in class.

Example. Is $p(x) = 2x - x^2$ an element of

$$\text{span}\left\{1 + x, x + x^2, 1 - x - x^2\right\}?$$

Solution.

Rephrase: is $p(x) = \overset{0+}{2x - x^2} = c_1(1+x) + c_2(x+x^2) + c_3(1-x-x^2)$?

$= (c_1 + c_3) + (c_1 + c_2 - c_3)x + (c_2 - c_3)x^2$

$\Rightarrow \begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 - c_3 = 2 \\ c_2 + c_3 = -1 \end{cases}$ SLE! $\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right].$

$\Rightarrow$ so yes!, $p(x) \in \text{span}\{1+x, x+x^2, 1-x-x^2\}$, and in particular:

$2x - x^2 = 3(1+x) - 4(x+x^2) - 3(1-x-x^2)$.

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Definition. Let V be a vector space. A subset $S \subseteq V$ is a *spanning set* for V when $V = \text{span } S$.

If $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$, then S is *linearly dependent* when there are $c_1, \ldots, c_k$ not all zero such that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$.

S is linearly independent when S is not linearly dependent.

(i.e. if $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$, then $c_1 = \ldots = c_k = 0$).

A *basis* for a vector space V is a linearly independent spanning set for V.

$\boxed{\textbf{Note.}}$ Since subspaces are vector spaces, the above definitions apply to subspaces.

Example. Find bases for each of the following subspaces.

1. $\operatorname{span}\{1+2x^2, -x, 2-3x+4x^2\} \subseteq P_2(\mathbb{R})$
2. $\{\vec{0}\}$ (where $\vec{0}$ is in some vector space V)
3. $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; a = 2d,\ a-b-c+d=0\right\} \subseteq M_{2,2}(\mathbb{R})$

Solution.

1) Do we have a spanning set? Yes! $S = \{1+2x^2, -x, 2-3x+4x^2\}$ is a spanning set for the subspace, by definition. Is it LI?

$c_1(1+2x^2) + c_2(-x) + c_3(2-3x+4x^2) = 0 + 0x + 0x^2)$

$\to (c_1+2c_3) + (-c_2)x + (2c_1+4c_3)x^2 = 0 + 0x + 0x^2$

$\to$ SLE! $\longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\Longrightarrow$ No, S is not LI.

free variable!

<u>But</u>, $S' = \{1+2x^2, -x\}$ <u>is</u> LI, and we know $2-3x+4x^2$ is a LC of polynomials in S', so $\operatorname{span} S = \operatorname{span} S'$. (ie. S' is a spanning set for $\operatorname{span} S$). Therefore, S' is a basis for $\operatorname{span} S$.

2) thm. $\vec{0}$ cannot be in any basis for $\{\vec{0}\}$, because $t\vec{0} = 0$ for all scalars t.

Idea: What is a linear combination of <u>no</u> vectors? By <u>convention</u>, we define an "empty" LC to be $\vec{0}$.

In this sense, the <u>empty set</u> $\{\}$ is a basis for $\{\vec{0}\}$.

3) First, let's use the defining restrictions to find a spanning set. $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = 2d,\ a-b-c+d=0\right\}$.

we get the following SLE:

$a - 2d = 0$
$a-b-c+d=0$ $\longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -3 \end{bmatrix}$.

$\underbrace{}_{c}\ \underbrace{}_{d} \to$ free variables!

Writing our solution in matrix form instead of a column vector, we have:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2d & -c+3d \\ c & d \end{bmatrix} = c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$.

So, we see that $\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right\}$ <u>spans</u> our subspace.

Is this set linearly independent?

<u>Option 1</u>: Yes! Look at the 0-1 pattern: for the same reason that we get LI vectors when solving SLE's, those matrices form a LI set.

<u>Option 2</u>: Set $c_1\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_2\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $\begin{cases} 2c_2 = 0 \\ -c_1+3c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases}$, from which we see that $c_1 = c_2 = 0$ is the only solution. So yes. $\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right\}$ is linearly independent,

Thus $\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a=2d,\ a-b-c+d=0\right\}$.

p. 250 **Theorem** (4.3.1)**.** *Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a spanning set for a vector space V.* Then every vector in V can be written as a <u>unique</u> LC of $\vec{v}_1, \dots \vec{v}_k$ if and only if B is linearly independent.

Proof.

($\Rightarrow$) Suppose that every vector in V is a <u>unique</u> LC of $\vec{v}_1, \dots, \vec{v}_k$. Then, if $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$, since $c_1 = \dots = c_k = 0$ is a solution, it must be the <u>only</u> solution, so B is linearly independent.

($\Leftarrow$) Suppose that B is LI. If $\vec{v} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k = d_1\vec{v}_1 + \dots + d_k\vec{v}_k$, then by subtracting we have $(c_1 - d_1)\vec{v}_1 + \dots + (c_k - d_k)\vec{v}_k = \vec{0}$. Since B is LI, we see that $c_i - d_i = 0$ for all i, so that $\vec{v}$ is written <u>uniquely</u> as a LC of the vectors in B.

or "B-coordinates" of $\vec{v}$

p. 264 **Definition.** Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for a vector space V. If $\vec{v} \in V$, the *coordinates* of $\vec{v}$ are the unique numbers $c_1, \dots, c_k$ such that $\vec{v} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$. The <u>B-coordinate vector</u> of $\vec{v}$ is

$$[\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k.$$

Example. We saw that $2 - 3x + 4x^2 = 2(1 + 2x^2) + 3(-x)$, so

$$[2 - 3x + 4x^2]_{S'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2 \quad (S' = \{1 + 2x^2, -x\} \text{ has two elements}).$$

MATH 122: strike out finite
"number" $\longrightarrow$ "cardinality"

p. 255 **Theorem** (4.3.4). Let V be a vector space w/ a finite basis. Then every basis for V has the same number of vectors.

Proof. Idea: Generalizing Theorem 2.3.4 to abstract vector spaces.

"$\square$"

p.255 **Definition.** Let V be a vector space. If V has a basis with finite size, then the *dimension* of V, denoted $\dim(V)$, is the size of any basis for V. If V has no finite basis, then we say that V has infinite dimension ($\dim(V) = \infty$).

↳ number of elements.

Example.

- $\dim(\mathbb{R}^3) = 3$; why? $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis for $\mathbb{R}^3$, w/ 3 vectors in it!
- $\dim(\operatorname{span}\{1+2x^2, -x, 2-3x+4x^2\}) = 2$, because $\{1+2x^2, -x\}$ is a basis for this subspace.
- $\dim(M_{m,n}(\mathbb{R})) = ?$ Elements of $M_{m\times n}(\mathbb{R})$ look like: m rows, n columns. "m×n matrices"

Eg. $m=2=n$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Looks like $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a spanning set for $M_{2\times 2}(\mathbb{R})$.

Is S linearly independent? Yes!

$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = c_4 = 0.$

$\Rightarrow S$ is a basis for $M_{2,2}(\mathbb{R}) \Rightarrow \dim(M_{2,2}(\mathbb{R})) = 4$. In general, $\dim(M_{m\times n}(\mathbb{R})) = mn$.

- In general, $\dim(\mathbb{R}^n) = n$, since $\{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$ containing n vectors.
- $\dim(P_n(\mathbb{R})) = \underline{n+1}$, not n, since a basis for $P_n(\mathbb{R})$ is $\{1, x, \dots, x^n\}$.
- $\dim(\{\vec{0}\}) = \underline{0}$, because the basis for $\{\vec{0}\}$ is $\{\ \}$ (the empty set), which has 0 vectors in it.
  - This fact explains the statement "points are zero-dimensional".
- $\dim\left(\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = 2d,\ a-b-c+d = 0\right\}\right) = 2$, since $\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for this space.
- What's the dimension of $P(\mathbb{R})$? Observe that if $B \subseteq P(\mathbb{R})$ is a finite set of polynomials, then there's a polynomial in B with the largest degree of any polynomial in B, say N. Consider x^{N+1}; x^{N+1} cannot be in $\operatorname{span} B$, since any polynomial in $\operatorname{span} B$ has maximum degree N and $\deg(x^{N+1}) = N+1 > N$.
  Thus, B cannot span $P(\mathbb{R})$, and so B is not a basis for $P(\mathbb{R})$.
  Therefore, $P(\mathbb{R})$ has no finite basis, so $P(\mathbb{R})$ has infinite dimensions.

Example. Revisiting Theorems 3.4.5/7/8: If $A \in M_{m,n}(\mathbb{R})$, then:

- $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A) = \dim(\mathrm{Row}(A))$
- $\dim(\mathrm{Null}(A)) = n - \mathrm{rank}(A)$
- $\dim(\mathrm{Null}(A^T)) = m - \mathrm{rank}(A)$

Example. Find a basis for $V = \{p(x) \in P_2(\mathbb{R}) \ : \ p(-1) = 0\}$ and extend it to a basis for $P_2(\mathbb{R})$. What is $\dim(V)$?

Solution. First, let's find a basis for V. If $\underset{\in V}{p(x) = a + bx + cx^2}$, then

$0 = p(-1) = a - b + c$. SLE!

$[\,1 \ \underbrace{-1 \ \ 1}_{\text{free variables}}\,] \longrightarrow$ solutions to SLE are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b-c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Back to $p(x)$: $p(x) = (b-c) + bx + cx^2 = b(1+x) + c(x^2-1)$.

$\Rightarrow V = \mathrm{span}\underbrace{\{1+x, x^2-1\}}_{\text{is this set LI?}}$.

Set $c_1(1+x) + c_2(x^2-1) = 0 \Rightarrow \begin{cases} c_1 - c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases} \longrightarrow c_1 = c_2 = 0$ is the only solution!

So yes! Thus $\{1+x, x^2-1\}$ is a basis for V. $\Rightarrow \dim(V) = 2$.

- How to extend to a basis for $P_2(\mathbb{R})$? Since $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$, we know that $\dim(P_2(\mathbb{R})) = 3$.

☆ • Try adding in a vector that's not in V! For example, $1 \notin V$.
Is $\{1+x, x^2-1, 1\}$ LI? Yes! Thus, if $\mathrm{span}\{1+x, x^2-1, 1\}$ is not $P_2(\mathbb{R})$, then we could add a fourth vector (or more) to get a basis for $P_2(\mathbb{R})$, which contradicts Thm. 4.3.4.

$\boxed{\text{Can be used as a sort of proof}}$

$\boxed{\text{Will be useful for other things in the course.}}$