June 24 (Lecture 13)

Overview: Today we'll figure out how to compute matrix inverses! And then we'll move beyond $\mathbb{R}^n$ and into the realm of *abstract vector spaces*.

Learning Goals:

- Correctly compute matrix inverses when possible.
- Identify abstract vector spaces (and subspaces thereof) and explain how the definitions apply in different contexts.

As you're getting settled:

- Homework 7 came out Tuesday evening (it was out on Brightspace before Crowdmark, whoops). (okay, more like ~3:30pm, sorry)
- Reflection will be available after class today! Due Friday night at 11:30 pm, as usual.
- Current events note.

Chapter 4

Abstract Vector Spaces

What do $\mathbb{R}^n$ and $M_{m,n}(\mathbb{R})$ have in common?

- Addition
- Multiplication (of some type? scalar)
- Row-Reduction*
- Subspaces (?)

both $\mathbb{R}^n$ and $M_{m,n}(\mathbb{R})$ have entrywise addition and scalar multiplication!

What other set of "things" also have those properties?

- Differential EQ's (solutions to), laplace transforms
- (Power) Series
- Polynomials
- Continuous functions on $\mathbb{R}$

Many objects we've seen before have "structure" like $\mathbb{R}^n$. we can add them together or multiply by scalars (real #'s), there's a "zero" thing, etc.

ex. "$p(y^2+5)$".

Notation for polynomials. "$p(x)$" or just "p", so we compose:

- $P(\mathbb{R})$ = set of all polynomials w/ real coefficients.
- $P_n(\mathbb{R})$ = set of all polynomials w/ maximum degree n.
  (degree of $p(x)$ = $\deg(p(x))$ = largest exponent in any term of $p(x)$).
  non-zero
  ex. $\deg(3+x^4+5x^3)=3$.
- If $p(x) = p_0 + p_1 x + \dots + p_n x^n$, $q(x) = q_0 + q_1 x + \dots + q_n x^n$, then

$$p(x)+q(x) = (p+q)(x) = (p_0+q_0) + (p_1+q_1)x + \dots + (p_n+q_n)x^n.$$

add $0x^k$ terms if necessary.

- $sp(x) = (sp)(x) = (sp_0) + (sp_1)x + \dots + (sp_n)x^n$.
- $p(x) = q(x)$ when $p_i = q_i$ for all i (all coefficients are equal).

(abstract)

p.240 **Definition.** A *vector space* (over $\mathbb{R}$) is a set V equipped with two operations, called *vector addition* and *scalar multiplication* and often denoted by $+$ and $\cdot$ (or just juxtaposition), such that for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $s, t \in \mathbb{R}$:

1. $\vec{x} + \vec{y} \in V$
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
4. There is $\vec{0} \in V$ (the *zero* vector) such that $\vec{x} + \vec{0} = \vec{x}$
5. For each $\vec{x}$ there is $\underline{-\vec{x}} \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
   additive inverse of $\vec{x}$.
6. $t\vec{x} \in V$
7. $s(t\vec{x}) = (st)\vec{x}$
8. $(s+t)\vec{x} = s\vec{x} + t\vec{x}$
9. $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$
10. $1\vec{x} = \vec{x}$

The elements of V are called *vectors*. (The textbook prefers to use bold notation, like $\mathbf{x}$ or $\mathbf{y}$.) Things like $c_1\vec{v}_1 + c_2\vec{v}_2$ are still called *linear combinations*.

* We saw the properties in
($\mathbb{R}^n$) Thm. 1.4.1 and
($M_{m,n}(\mathbb{R})$) Thm 3.1.1

- $\mathbb{R}^n$! (Theorem 1.4.1)
- $M_{m,n}(\mathbb{R})$ (Theorem 3.1.1)

Example.

- $P_n(\mathbb{R})$! (4) What is "$\vec{0}$" $\in P_n(\mathbb{R})$? Well, if $0 = 0 + 0x + \dots + 0x^n$ (+ ....), then
  $p(x) + 0 = (p_0 + 0) + (p_1 + 0)x + \dots + (p_n + 0)x^n = p(x)$.
  $\Rightarrow \vec{0}$ = the zero polynomial. (Note: $0 \in P_n(\mathbb{R})$!)

- $P(\mathbb{R})$! (1) If $p(x), q(x) \in P(\mathbb{R})$, is $(p+q)(x) \in P(\mathbb{R})$? Yes! In particular, if $\deg(p(x)) = k < m = \deg(q(x))$, then...
  $(p+q)(x) = (p_0 + q_0) + \dots + (p_k + q_k)x^k + (0 + q_{k+1})x^{k+1} + \dots + q_m x^m \in P(\mathbb{R})$.

- $D = \{ p(x) \in P(\mathbb{R}) : \deg(p(x)) = 42 \}$ $1 + x^{42}, 1 - x^{42} \in D$, but $(1+x^{42}) + (1-x^{42}) = 2 \notin D$ No, not a vector space.
  degree 0!
- $Z = \{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{Z} \}$ "Integers" $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in Z$, but $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \notin D$ No, not a vector space (over $\mathbb{R}$).

Example. The following vector space probably looks *very* wrong.

- Let $E = \mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$.
  "$\mathcal{E}$ for exponents!"
- For two elements x and y of E, define $x \oplus y$ to be xy.
- For a real number α and $x \in E$, define αx to be x^α.
  or $\alpha \odot x$

$\boxed{8}$

$\boxed{3}$

$\boxed{4}$ What is $\vec{0} \in \mathcal{E}$?

Idea: When looking at exponents of positive real numbers, multiplication and exponentiation look a lot like "addition" and "multiplication" (in $\mathbb{R}^1$).

p. 243 **Theorem** (4.2.1)**.** *Let V be a vector space. Then for all $\vec{x} \in V$ and $t \in \mathbb{R}$:*

1. $0\vec{x} = \vec{0}$
2. $(-1)\vec{x} = -\vec{x}$ (th additive inverse of $\vec{x}$)
3. $t\vec{0} = \vec{0}$.

Proof.

Of 2.): We use th axioms!

$$(-1)\vec{x} \overset{(4)}{=} (-1)\vec{x} + \vec{0} \overset{(5)}{=} (-1)\vec{x} + \vec{x} + (-\vec{x}) \overset{(10)}{=} (-1)\vec{x} + 1\vec{x} + (-\vec{x}) \overset{(8)}{=} (-1+1)\vec{x} + (-\vec{x}) =$$

$$0\vec{x} + (-\vec{x}) \underset{\text{Part 1.)}}{=} \vec{0} + (-\vec{x}) \underset{(4)}{=} -\vec{x}$$

"$\blacksquare$"

Example. "$-\vec{x}$" in different vector spaces:

- $\mathbb{R}^n$: "$-\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$" $= \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}$.
- $M_{m,m}(\mathbb{R})$: "$-A$" $=$
- $P(\mathbb{R})$: "$-p(x)$" $= -a_0 + (-a_1)x + \dots + (-a_n)x^n$ if $p(x) = a_0 + \dots + a_n x^n$.

p. 244 **Definition.** Let V be a ~~sub~~ Vector space. A set $W \subseteq V$ is a *subspace* of
(df. p. 51) V when W is non-empty and for all $\vec{x}, \vec{y} \in W$, $s, t \in \mathbb{R}$, we have $s\vec{x} + t\vec{y} \in W$. ($W$ is closed under LC's).

checkable!

Equivalently, W is a subspace of V when W is a subset of V that is also a vector space under th same operations as V.

Example. Show that $U = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;:\; a,b,c \in \mathbb{R} \right\}$ is a subspace of $M_{2,2}(\mathbb{R})$.

Solution. Is U non-empty? Yes: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U$ ($a=b=c=0$). ✓

Is U closed under LC's? Let $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \in U$, $s,t \in \mathbb{R}$.

$s\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + t\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} sa & sb \\ 0 & sc \end{bmatrix} + \begin{bmatrix} td & te \\ 0 & tf \end{bmatrix} = \begin{bmatrix} sa+td & sb+te \\ 0 & sc+tf \end{bmatrix} \in U.$

$\Rightarrow$ Yes, U is closed under LC's, and therefore U is a subspace.

Example. Show that $C = \left\{ a + bx^3 \;:\; a,b \in \mathbb{R} \right\}$ is a subspace of $P_3(\mathbb{R})$.

Solution.

Is C non-empty? Yes! $0 = 0 + 0x^3 \in C$, as $0 \in \mathbb{R}$.

Is C closed under LC's? Let $a+bx^3, c+dx^3 \in C$, $s,t \in \mathbb{R}$.

Then: $s(a+bx^3) + t(c+dx^3) = \underbrace{(sa+tc)}_{\in \mathbb{R}} + \underbrace{(sb+td)}_{\in \mathbb{R}}x^3 \in C$!

So YES, C is closed under LC's, and thus C is a subspace of $P_3(\mathbb{R})$.

Example. Why isn't $\left\{ ax + x^2 \;:\; a \in \mathbb{R} \right\}$ a subspace of $P_2(\mathbb{R})$?

- It's not that it's empty.
- Not closed under scalar multiplication! ex. $x+x^2$ is in the set, but $2(x+x^2) = 2x+2x^2$ is not.

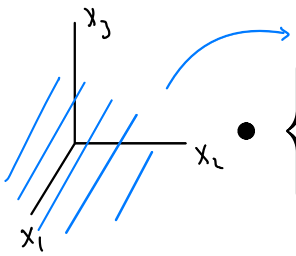
(• Also not closed under addition)

Example. Which of the following are subspaces?

- V, as a subset of any vector space V Yes! By definition!

- $\{\vec{0}\}$, as subset of any vector space V Yes! By Thm. 4.2.1, basically.

- $\mathbb{R}^2$, as a subset of $\mathbb{R}^3$ No! Why? $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.
  If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, then $\vec{x} \notin \mathbb{R}^3$.



- $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$, as a subset of $\mathbb{R}^3$ Yes!
  (It looks like $\mathbb{R}^2$, but now it is actually a subset of $\mathbb{R}^3$.)

p. 249 **Definition.** Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space V. The *span* of S, denoted $\operatorname{span} S$, is the set

$$\{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \ : \ c_1, \ldots, c_k \in \mathbb{R}\}.$$

p. 246 **Theorem** (4.2.2)**.** *Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space V. Then* $\operatorname{span} S$ *is a subspace of V.*

Proof.
First, $\vec{0} = 0\vec{v}_1 + \dots + 0\vec{v}_n \in \operatorname{span} S$, so $\operatorname{span} S$ is non-empty!
Then, let $\underbrace{a_1\vec{v}_1 + \dots + a_k\vec{v}_k}_{\vec{a}}, \underbrace{b_1\vec{v}_1 + \dots + b_k\vec{v}_k}_{\vec{b}} \in \operatorname{span} S$, and $s, t \in \mathbb{R}$. We have:

$$s\vec{a} + t\vec{b} = (sa_1)\vec{v}_1 + \dots + (sa_k)\vec{v}_k + (tb_1)\vec{v}_1 + \dots + (tb_k)\vec{v}_k = (sa_1 + tb_1)\vec{v}_1 + \dots + \vec{v}_k \in \operatorname{span} S.$$

So $\operatorname{span} S$ is closed under LC's!
Thus it is a subspace of V. ∎