June 21 (Lecture 12)

Overview: Today we will talk about "invertible" matrices and how to compute matrix inverses!

Learning Goals:

- Define invertibility for matrices and check if a matrix is invertible.
- Correctly compute matrix inverses when possible.

As you're getting settled:

- Just me from here (no more help from Elizabeth :( )

- Test 1 marking is done! You should have your marks now; please be sure to check the marking and let me know if you have questions or concerns.
- Homework 6 is due Tuesday night at 11:30 pm, as usual.
- Homework 7 to come out on Tuesday during the day!

Let $A \in M_{n,n}(\mathbb{R})$.

Finding inverses of $n \times n$ matrices: We want to find B so $AB = I_n$.

Look at the columns: if $B = [B_1 \cdots B_n]$, then

$$AB = I_n \qquad \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$[AB_1 \,|\, \cdots \,|\, AB_n] = [\vec{e}_1 \,|\, \cdots \,|\, \vec{e}_n].$$ If such B exists, then the columns of B solve the SLEs $\underline{A\vec{x} = \vec{e}_i}$!

To solve all of those SLEs simultaneously, use a big augmented matrix!

$$[A \,|\, \vec{e}_1 \cdots \vec{e}_n] = [A \,|\, I_n] \xrightarrow{\text{row ops}} [C \,|\, J] \text{ in RREF.}$$

Two Cases:

1) $C = I_n$. $\left[\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \middle| \; \underbrace{|}_{\text{solution to } A\vec{x} = \vec{e}_i} \; | \cdots | \right]$, so A is invertible, w/ $A^{-1} = J$.

2) $C \neq I_n$: Then $\operatorname{rank}(A) < n$, so A is not invertible by Theorem 3.5.2.

Algorithm (3.5.1). Finding a matrix inverse for $A \in M_{n,n}(\mathbb{R})$.

1) Row-reduce $[A \,|\, I_n]$ to RREF.

2) If the RREF is $[I_n \,|\, J]$, then $A^{-1} = J$.

3) If not, then A is not invertible. (i.e. $\operatorname{RREF}(A) \neq I_n$)

Example. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$.

Solution.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -4 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 4R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -6 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right]$$

I_3 ✓ $\qquad A^{-1}$

(matches what we've already seen)!

> e.g. If $[A \mid I_2]$ row-reduced to *, then A is not invertible.
>
> ex. But this is not I_2.
>
> * $\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$
>
> RREF

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Theorem (3.5.4). *Let $A \in M_{n,n}(\mathbb{R})$. The following are equivalent* (TFAE) *(i.e. one is true if and only if all of them are true):*

1. A is invertible.
2. $\text{rank}(A) = n$.
3. $\text{RREF}(A) = I_n$.
4. For all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution.
5. $\text{Null}(A) = \{\vec{0}\}$.
6. $\text{Col}(A) = \mathbb{R}^n$.
7. $\text{Row}(A) = \mathbb{R}^n$.
8. $\text{Null}(A^T) = \{\vec{0}\}$.
9. A^T is invertible.
10. (See this later!)

We've seen most of this theorem before! So it's just a matter of putting it together.

#4

Example. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$, solve $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$.

Solution. We know $A^{-1} = \begin{bmatrix} 3 & -6 & -1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$. So, if $A\vec{x} = \vec{b}$, then we can write $\underline{\vec{x}} = I_3\vec{x} = A^{-1}A\vec{x} = \underline{A^{-1}\vec{b}}$.

>> For $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 3 & -6 & -1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 9 \end{bmatrix}$,

for $\vec{b} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$, $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 3 & -6 & -1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 22 \\ -2 \\ -15 \end{bmatrix}$.

Example. Let $B = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Determine whether or not B is invertible, and compute B^{-1} if it is.

$$[B \mid I_4] = \left[\begin{array}{cccc|cccc} 1 & 2 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_4]{R_1 - R_4} \left[\begin{array}{cccc|cccc} 0 & 2 & -3 & 1 & 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & -2 & 2 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$R_3 = 2R_2$

We can see that when $[B \mid I_4]$ is put into RREF,

It will not look like $[I_4 \mid J]$. (since $\text{rank}(B) < 4$). Thus B is not invertible.