June 17 (Lecture 11)

Overview: Today we'll get started on subspaces associated to linear maps and matrices (more connections between all of these "linear" things), and then talk about "invertible" matrices.

Learning Goals:

- Define and compute subspaces associated to linear maps and matrices.
- Define invertibility for matrices and check if a matrix is invertible.

As you're getting settled:

- Test 1 marking to be finished today /tomorrow (sorry it's not quite done yet); when you do get your marks back, please be sure to check the marking and let me know if you have questions or concerns.
- Homework 6 is due Tuesday night, of course.
- This week's Reflection will be available a couple hours after class, due Friday night as usual. This week it's a bit of a "halfway survey" on the class; thanks in advance for providing feedback!

Section 3.5

* "multiplicative inverse" of a

Invertible Matrices

Motivation. Is there an analogue to solving "$ax = b$" for $A\vec{x} = \vec{b}$?

For $ax=b$: multiply by $\frac{1}{a}^{*}$ to get $x = \frac{1}{a}b = \frac{b}{a}$.

only if $a \neq 0$. ← condition on a

What is A^{-1}, and when does it exist?

p. 207 **Definition.** Let A be an $n \times n$ matrix (so A is square). We say that A is *invertible* when there exists an $n \times n$ matrix B such that
★ $AB = BA = I_n$. In this case, B is called the inverse of A, denoted by $B = A^{-1}$.

Example. Consider the following 2×2 matrices:

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 7 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}, \; E = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

First A: $AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, and $BA = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

$\Rightarrow A$ is invertible, and $A^{-1} = B = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$.

Let $F = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$ be some 2×2 matrix.

B: From ↗, $BA = AB = I_n$★, so B is invertable, with $B^{-1} = A$.

C: $FC = \left[F\begin{bmatrix} 1 \\ 7 \end{bmatrix} \middle| F\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \neq I_2$. $\Rightarrow C$ is not invertable.

D: $DF = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} * & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \neq I_2$. $\Rightarrow D$ is not invertible.

E: Let $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Then if E were invertible $\vec{x} = I\vec{x} = E^{-1}E\vec{x} = E^{-1}\left(2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) = E^{-1}\vec{0} = \vec{0}$ ($\vec{x} \in Null(B)$)

Contradiction! Thus E is not invertible. ¨̂

Note. What do C, D, E have in common? They all do not have rank $= 2$.

Example. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. When is A invertible, and in that case what is A^{-1}? Observe the following matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- When $ad-bc \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$! (and A is invertible)

- When $ad-bc = 0$, A is <u>not</u> invertible. If it <u>were</u> invertible, then $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}A\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}\,0_{2\times 2} = 0_{2\times 2}$, which is not the case unless $A = 0_{2\times 2}$, which is not invertible (by inspection).

Example. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Let $B = \begin{bmatrix} 3 & -6 & -1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & -6 & -1 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Similarly, $BA = I_3$, so A is invertible, with $A^{-1} = B$.

How can we determine if a larger matrix, 3×3 or bigger, is invertible?

p. 208 **Theorem** (3.5.1). If A is invertible, then A^{-1} is unique.

Proof. Suppose B, C are matrix inverses for A. Then $AC = I = BA$. Then

$B = BI = B(AC) = (BA)C = IC = C$. (Thus $B = A^{-1} = C$)

$\blacksquare$

p. 208 **Theorem** (3.5.2). If $A, B \in M_{n,n}(\mathbb{R})$ such that $AB = I_n$, then $BA = I_n$ and $B = A^{-1}$. Moreover, $\text{rank}(A) = n = \text{rank}(B)$.

Proof. First, if $B\vec{x} = \vec{0}$, then $\vec{x} = I\vec{x} = AB\vec{x} = A\vec{0} = \vec{0}$, and thus $\text{rank}(B) = n - 0 = n$.

Now, if $\vec{y} \in \mathbb{R}^n$, then find $\vec{x} \in \mathbb{R}^n$ such that $B\vec{x} = \vec{y}$. Then: $BA\vec{y} = BAB\vec{x} = B(AB)\vec{x} = B\vec{x} = \vec{y} = I\vec{y}$. By Thm 3.1.5 $BA = I_n$.

Finally, this proof works for $A\vec{x} = \vec{0}$ also.

$\blacksquare$

p. 209 **Theorem** (3.5.3). *Let A and B be invertible $n \times n$ matrices. Let $t \in \mathbb{R}$ be non-zero.*

1. $(tA)^{-1} = t^{-1}A^{-1}$.
2. $(AB)^{-1} = B^{-1}A^{-1}$. (note the order)!
3. $(A^T)^{-1} = (A^{-1})^T$.

Proof. For (#2): $(AB)(B^{-1}A^{-1}) = A(\underbrace{BB^{-1}}_{I_n})A^{-1} = AI_nA^{-1}$

By theorem 3.5.2,

AB is invertible, with inverse $B^{-1}A^{-1}$.

$\square$