June 14 (Lecture 10)

Overview: The bulk of today will be spent on looking at how various geometrical operations in $\mathbb{R}^2$ and up can be seen as linear maps! Then we'll get started on subspaces associated to linear maps and matrices (more connections between all of these "linear" things).

Learning Goals:

- Identify certain geometric transformations as linear maps.
- Compute subspaces associated to linear maps and matrices.

As you're getting settled:

- I hope Test 1 went well for everyone! Marking is in progress, still. :)
- Homework 5 is due Tuesday night, 11:30 pm Pacific as usual.
- Homework 6 will be out Tuesday during the day.

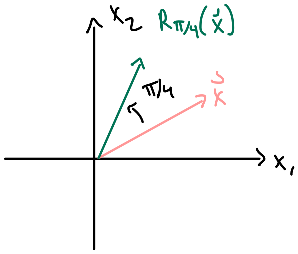
3.4 Subspaces Associated to Linear Maps/Matrices

- Non-homog. SLE $A\vec{x} = \vec{b}$: "Is $\vec{b}$ the output of f_A?" $f_A(\vec{x}) = \vec{b}$?
- Homog. SLE $A\vec{x} = \vec{0}$: "Does f_A send $\vec{x}$ to $\vec{0}$?" $f_A(\vec{x}) = \vec{0}$?

p.192 **Definition.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The *range* (or *image*) of L is $\text{range}(L) = \{L(\vec{x}) : \vec{x} \in \mathbb{R}^n\}$.

p.193 The *nullspace* (or *kernel*) of L is $\text{Null}(L) = \{\vec{x} \in \mathbb{R}^n : L(\vec{x}) = \vec{0}\}$.

Example. Let $R_{\pi/4} : \mathbb{R}^2 \to \mathbb{R}^2$ be CCW rotation by $\pi/4$.

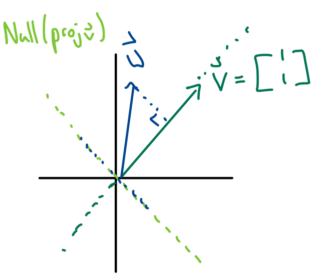


$\text{range}(R_{\pi/4}) = \{R_{\pi/4}(\vec{x}) : \vec{x} \in \mathbb{R}^2\} = \mathbb{R}^2! \subseteq \mathbb{R}^m$.

Note that $\vec{x} = R_{\pi/4}(\underbrace{R_{-\pi/4}(\vec{x})}_{\vec{y}}) = R_{\pi/4}(\vec{y}) \subseteq \mathbb{R}^n$.

$\text{Null}(R_{\pi/4}) = \{\vec{x} \in \mathbb{R}^2 : R_{\pi/4}(\vec{x}) = \vec{0}\} = \{\vec{0}\}$ (Rotating does not change the norm of $\vec{x}$!)

Example. What are Range($\text{proj}_{\vec{v}}$) and Null($\text{proj}_{\vec{v}}$), if $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?



$\text{proj}_{\vec{v}}(\vec{x}) = \frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = c\vec{v}$, and by linearity all scalar multiples of $\vec{v}$ are obtained. Thus, $\text{Range}(\text{proj}_{\vec{v}}) = \text{span}\{\vec{v}\}$.

For $\text{proj}_{\vec{v}}(\vec{x}) = \vec{0}$, we require $\frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \vec{0}$, or $(\vec{x}\cdot\vec{v})\vec{v} = \vec{0}$. Since $\vec{v} \neq 0$, the only way this occurs is when $\vec{x}\cdot\vec{v} = 0$. Solving the SLE $\Rightarrow \text{Null}(\text{proj}_{\vec{v}}) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

p.193 **Theorem** (3.4.1)**.** *If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $L(\vec{0}) = \vec{0}$.*

Proof. $L(\vec{0}) = L(0\cdot\vec{0}) \underset{\text{linearity}}{=} 0\cdot L(\vec{0}) = \vec{0}$. ∎

p. 193-194 **Theorem.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, with standard matrix $[L]$.*

(3.4.2) Range (L) is a subspace of $\mathbb{R}^m$.

(3.4.3) Null(L) is a subspace of $\mathbb{R}^n$.

(3.4.4) $\vec{x} \in$ Range(L) if and only if $\vec{x}$ is a LC of the columns of $[L]$.

(3.4.6) $\vec{x} \in$ Null(L) if and only if $[L]\vec{x} = \vec{0}$.

Proof. For 3.4.2, 3.4.3. Range(L) and Null(L) are non-empty, by 3.4.1. If $s,t \in \mathbb{R}$, $L(\vec{x})$ and $L(\vec{y}) \in$ Range(L), then:

$sL(\vec{x}) + tL(\vec{y}) = L(s\vec{x} + t\vec{y}) \in$ Range(L), by linearity of L! ✓

If $s,t \in \mathbb{R}$, $\vec{x}, \vec{y} \in$ Null(L):

$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y}) = s\vec{0} + t\vec{0} = \vec{0}$. ✓

p. 195, 198, 200 **Definition.** Let $A \in M_{m,n}(\mathbb{R})$. The *four fundamental subspaces* associated to A are:

- the *column space* of A, denoted Col(A):

  $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\} = \text{span}\{\text{columns of } A\} = \text{Range}(f_A) \subseteq \mathbb{R}^m$.
- the *nullspace* of A, denoted Null(A):

  $\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\} = \text{Null}(f_A) \subseteq \mathbb{R}^n$.
- the *row space* of A, denoted Row(A):

  $\text{Span}\{\text{rows of } A\} = \text{Col}(A^T) \subseteq \mathbb{R}^n$.
- the *left nullspace* of A:

  $\{\vec{x} \in \mathbb{R}^m : \underset{1\times m}{\vec{x}^T}\, \underset{m\times n}{A} = \vec{0}^T\} = \text{Null}(A^T) \subseteq \mathbb{R}^m$.

Note.

All four of those sets are subspaces!

Example. Let $A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 2 & 4 & -2 & 0 & -2 \\ 2 & 5 & -4 & -7 & -5 \\ -1 & -4 & 6 & 18 & 9 \\ 1 & 0 & 4 & 18 & 7 \end{bmatrix} \end{array}$. Compute each of the four fundamental subspaces for A.

Col(A): what is span{cols of A}? Row-reduce A and find a basis!

$$\begin{bmatrix} 2 & 4 & -2 & 6 & -2 \\ 2 & 5 & -4 & -7 & -5 \\ -1 & -4 & 6 & 18 & 9 \\ 1 & 0 & 4 & 18 & 7 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B, \text{ in RREF.}$$

The 4th and 5th columns are LC's of the first 3, and moreover, the set of the first 3 vectors is LI.

So.... $\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 6 \\ 4 \end{bmatrix} \right\}$ is a basis for Col(A).

Null(A): What are the solutions to $A\vec{x} = \vec{0}$?

Again: $A \xrightarrow{\text{row ops}} B = RREF(A)$. x_4, x_5 are free variables! $\vec{x} = \begin{bmatrix} -2x_4 + x_5 \\ -x_4 - x_5 \\ -4x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ -1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

So $\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ -1 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Row(A): Row ops replace rows with LC's of rows. So, performing row ops does not change the row space!

$\Rightarrow \text{Row}(A) = \text{Row}(B) = \text{Col}(B^T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 2 \end{bmatrix} \right\}$.

(or $\text{col}(A^T)$)

$\text{Null}(A^T)$: Solve $A^T\vec{y} = \vec{0}$: $A^T \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C$, so $\text{Null}(A^T) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

p. 197, 199, 200 **Theorem.** *Let $A \in M_{m,n}(\mathbb{R})$. The number of vectors in a basis for each of the fundamental subspaces is:*

(3.4.5) $\mathrm{Col}(A)$: $\mathrm{rank}(A)$.

(3.4.7) $\mathrm{Null}(A)$: $n - \mathrm{rank}(A)$.

(3.4.8) $\mathrm{Row}(A)$: $\mathrm{rank}(A)$.

- $\mathrm{Null}(A^T)$: $m - \mathrm{rank}(A)$.

Note:

$\mathrm{rank}(A^T)$ = # vectors in a basis for $\mathrm{col}(A^T)$
= .... $\mathrm{Row}(A)$
= $\mathrm{rank}(A)$

Proof.

$\mathrm{rank}(A)$ = # leading ones in RREF for A → basis for $\mathrm{Col}(A)$.
= # non-zero rows in RREF for A → basis for $\mathrm{Row}(A)$.

$n - \mathrm{rank}(A)$ = # of free variables in sol'n set to $A\vec{x} = \vec{0}$ → basis for $\mathrm{Null}(A)$.

$m - \mathrm{rank}(A)$ = # of free variables in sol'n set to $A^T\vec{y} = \vec{0}$ → basis for $\mathrm{Null}(A^T)$.
$= m - \mathrm{rank}(A^T)$

∎

p 201 **Theorem** (3.4.9, Rank-Nullity Theorem)**.** *Let $A \in M_{m,n}(\mathbb{R})$. If there are k vectors in a basis for* $\mathrm{Null}(A)$, *then* $\mathrm{rank}(A) + k = n$.

Proof. $\mathrm{rank}(A) + k \overset{3.4.7}{=} \cancel{\mathrm{rank}(A)} + n - \cancel{\mathrm{rank}(A)} = n$. ∎

Example. Suppose that B is a non-zero 3×7 matrix. What can you say about the number of vectors in a basis for $\mathrm{Col}(B)$ or $\mathrm{Null}(\cancel{A})$? B

$$B = \begin{array}{cc} & 7 \\ 3 & \begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \end{bmatrix} \end{array}$$

- $\mathrm{rank}(B)$ + # vectors in a basis for $\mathrm{Null}(B) = 7$
- since B has 3 rows, $\mathrm{rank}(B) \leq 3$. (also ≥ 1).

so $\mathrm{rank}(B) \in \{1,2,3\}$, so # basis vectors for $\mathrm{col}(B) \in \{1,2,3\}$,
and # basis vectors for $\mathrm{Null}(B) \in \{4,5,6\}$.

Note.

"# basis vectors for a subspace"
→ def'n of dimension of a subspace, to see later!

Summary. Let A be an $m \times n$ matrix, B the RREF of A, and C the RREF of A^T.

Subset	Notation	Abstract description	Concrete description
Column space	$\mathrm{Col}(A)$ $\subseteq \mathbb{R}^m$	$\vec{b}$ st. $A\vec{v} = \vec{b}$ for some $\vec{v} \in \mathbb{R}^n$	Span of the columns of A corresponding to columns of B with leading ones.
Row space	$\mathrm{Row}(A) = \mathrm{Col}(A^T)$ $\subseteq \mathbb{R}^n$	$\vec{c}$ st. $A^T\vec{v} = \vec{c}$ $\iff \vec{v}^T A = \vec{c}^T$	Span of the non-zero rows of B / columns of B^T.
Null space	$\mathrm{Null}(A)$ $\subseteq \mathbb{R}^n$	$\vec{x}$ st. $\boxed{A\vec{x} = \vec{0}}$	(span of) solutions vectors to $B\vec{x} = \vec{0}$.
Left null space	$\mathrm{Null}(A^T)$ $\subseteq \mathbb{R}^m$	$\vec{y}$ st. $A^T\vec{y} = \vec{0}$ $\iff \boxed{\vec{y}^T A = \vec{0}^T}$ "left"	(span of) solution vectors to $C\vec{y} = \vec{0}$.

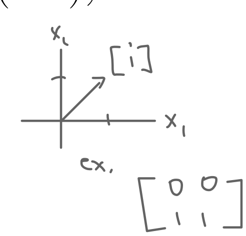
Theorem (3.4.10, Fundamental Theorem of Linear Algebra). *Let $A \in M_{m,n}(\mathbb{R})$. We have:*

* "Orthogonal Compliments" textbook

1. $\mathrm{Null}(A) = \{\vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{r} = 0$ for all $\vec{r} \in \mathrm{Row}(A)\}$.
2. $\mathrm{Null}(A^T) = \{\vec{y} \in \mathbb{R}^m : \vec{y} \cdot \vec{c} = 0$ for all $\vec{c} \in \mathrm{Col}(A)\}$.
3. *If B_1 is a basis for* $\mathrm{Row}(A)$ *and B_2 is a basis for* $\mathrm{Null}(A)$, *then* $B_1 \cup B_2$ is a basis for $\mathbb{R}^n$.
4. *If B_3 is a basis for* $\mathrm{Col}(A)$ *and B_4 is a basis for* $\mathrm{Null}(A^T)$, *then* $B_3 \cup B_4$ is a basis for $\mathbb{R}^m$.

"$\cup$": set union

• put all elements together.



Picture.