June 7 (Lecture 9)

Overview: Today we'll define linear maps and see how they are related to matrices and matrix multiplication, with some examples.

Learning Goals:

- Precisely define and check for linear maps.
- Correctly compute the standard matrix for a linear map.
- Identify certain geometric transformations as linear maps.

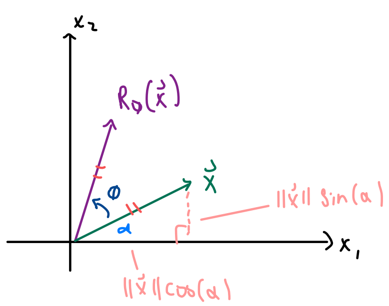
As you're getting settled:

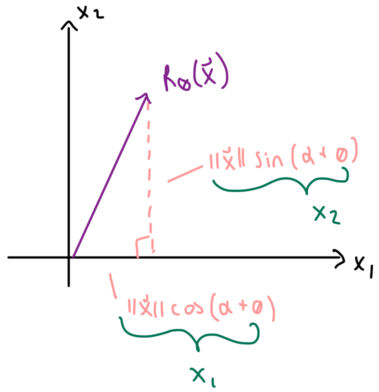
- Test 1 on Thursday! Please see the Announcement posted on Brightspace for details.
- In addition to my normal office hours this week, Elizabeth also has an office hour! Tuesday, 2-2:50 pm. Details can be found on the office hours page on Brightspace.
- HW 5 is probable (though shorter / more computational)

Geometrical Transformations

Example. Consider the map $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector counter-clockwise by θ (radians). Is R_θ linear? If so, what is $[R_\theta]$?

Let's find a formula for R_θ!





Using our diagrams, we have:

$$\vec{x} = \begin{bmatrix} \|\vec{x}\| \cos(\alpha) \\ \|\vec{x}\| \sin(\alpha) \end{bmatrix},$$

addition trig identities

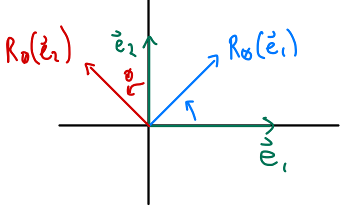
$$R_\theta(\vec{x}) = \begin{bmatrix} \|\vec{x}\| \cos(\alpha+\theta) \\ \|\vec{x}\| \sin(\alpha+\theta) \end{bmatrix} = \|\vec{x}\| \begin{bmatrix} \cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta) \\ \sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta) \end{bmatrix} = \begin{bmatrix} x_1 \cos(\theta) - x_2 \sin(\theta) \\ x_2 \cos(\theta) + x_1 \sin(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

R_θ is thus a matrix map, hence linear (Thm. 3.2.2), and

$$[R_\theta] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

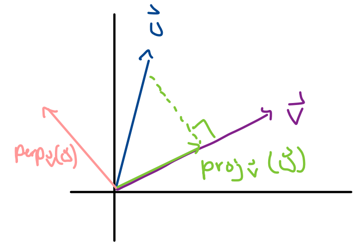
Picture:



Other geometrical transformations:

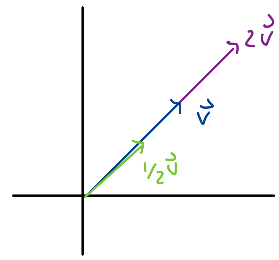
- Projections/perpendicular parts: in $\mathbb{R}^n$!

$$[\mathrm{proj}_{\vec{v}}] = \left[\cdots \ \frac{\vec{v}\cdot\vec{e}_i}{\|\vec{v}\|^2}\vec{v} \ \cdots\right] = \begin{array}{c} \text{ith column!} \\ \left[\cdots \ \vdots \ \frac{v_i}{\|\vec{v}\|^2}\vec{v} \ \vdots \ \cdots\right] \end{array}$$



$$[\mathrm{perp}_{\vec{v}}] = \left[\cdots \ \vdots \ \vec{e}_i - \frac{\vec{v}\cdot\vec{e}_i}{\|\vec{v}\|^2}\vec{v} \ \vdots \ \cdots\right] \quad \left(= [\mathrm{id}_n] - [\mathrm{proj}_{\vec{v}}] \right)$$

- Contractions/dilations: $T = \underbrace{t}_{\text{scale factor}} \cdot \mathrm{id}_n$ in $\mathbb{R}^n$



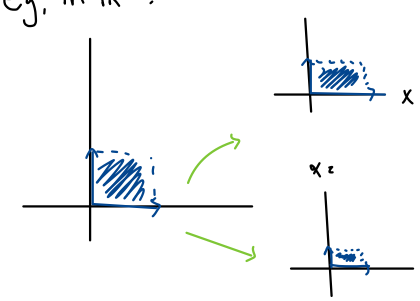
$$[T] = [\cdots T(\vec{e}_i) \cdots] = [\cdots t\vec{e}_i \cdots]$$
$$= \begin{bmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{bmatrix} = t I_n.$$

- Stretches/shrinks:

$$[S] = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}$$

(or "s in the (ii)th entry" instead of 1 for $\mathbb{R}^n$)

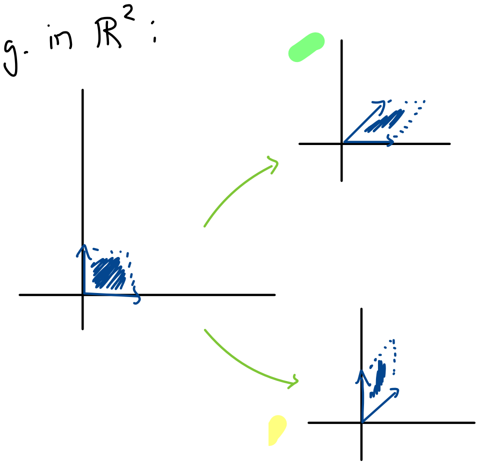
eg. in $\mathbb{R}^2$:



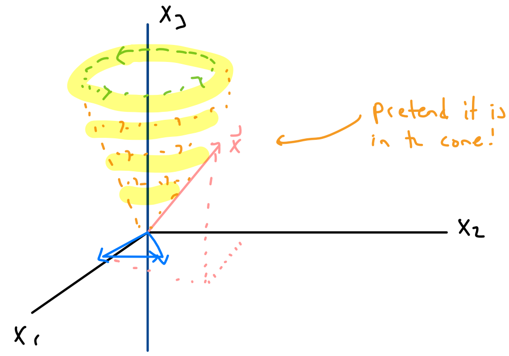
- Shears:

$$[S] = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

eg. in $\mathbb{R}^2$:



- Rotations in $\mathbb{R}^3$: Rotating around an <u>axis</u>!



eg. rotating around th x_3-axis.

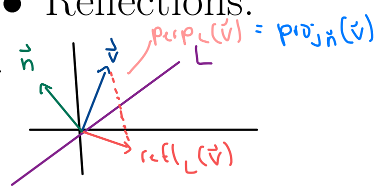
Turns out that this map is linear!

With standard matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(Similar for rotation around x_1, x_2-axes).

- Reflections:

<u>First, in $\mathbb{R}^2$:</u>

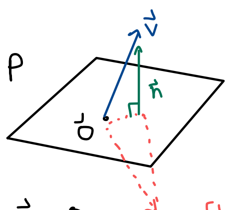


LC of linear maps!
$\Rightarrow$ linear.

$\Rightarrow$ <u>$\text{refl}_L(\vec{v}) = \vec{v} - 2\,\text{proj}_{\vec{n}}(\vec{v})$.</u>

$\Rightarrow [\text{refl}_L] = [id_2] - 2[\text{proj}_{\vec{n}}]$ (Theorem 3.2.4)

<u>In $\mathbb{R}^n$:</u>



$\vec{n}\cdot\vec{x} = 0$

$\text{refl}_P(\vec{v}) = \vec{v} - 2\,\text{proj}_{\vec{n}}(\vec{v})$. <u>Same thing!</u>

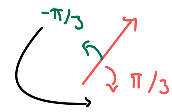
$[\text{refl}_P] = \underbrace{[id_n]}_{I_n} - 2[\text{proj}_{\vec{n}}]$

<u>Identity map</u>

Note: Needs to be through th origin!

Example. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that reflects a vector over the x_2-axis and then rotates clockwise by $\pi/3$ radians. Find $[L]$.

Solution. L is a composition of linear maps! =



$L = R_{-\pi/3} \circ \text{refl}_{x_2\text{-axis}}$, so ...

$$[L] = [R_{-\pi/3}][\underbrace{\text{refl}_{x_2\text{-axis}}}_{\vec{n} = \vec{e}_1}] = \begin{bmatrix} \cos(-\pi/3) & -\sin(-\pi/3) \\ \sin(-\pi/3) & \cos(-\pi/3) \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2[\text{proj}_{\vec{e}_1}] \right)$$

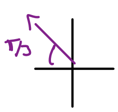
$$= \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

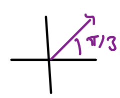
$$= \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}. \quad \left(= \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) \end{bmatrix} \right)$$

*

After the fact:

$L(\vec{e}_1) = R_{-\pi/3}(-\vec{e}_1) = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$, $L(\vec{e}_2) = R_{-\pi/3}(\vec{e}_2) = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$.





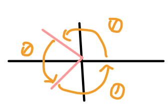
Example. Find a matrix $A \in M_{2,2}(\mathbb{R})$ such that $\underline{A^3 = I_2}$. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution. We don't have a "cube root" for matrices.

- $A = I_2$! Then $A^3 = I_2[I_2 I_2] = I_2 I_2 = I_2$. ✓

  $I_2 = [\text{id}_1(\vec{e}_1) \ \ \text{id}_2(\vec{e}_2)] = [\text{id}_2]$.

- Is there R_θ such that $R_\theta^{\,3} = \text{id}_2$?



  Yes! if $3\theta = 2\pi$, then setting $\theta = 2\pi/3$ give $R_{2\pi/3} = \text{id}_2$, so $[R_{2\pi/3}]^3 = [\text{id}_2] = I_2$. [not the only rotation map with this property]