June 3 (Lecture 8)

Overview: More matrix operations! We'll start trying to define multiplication of matrices, and time-permitting we'll move on to talking about linear maps.

Learning Goals:

- Correctly perform basic computations with matrices, including matrix multiplication.
- Precisely define and check for linear maps.

As you're getting settled:

- Test 1 next Thursday! Please see the Announcement posted on Brightspace for details.
- Please reach out and ask for help instead of violating academic integrity. (Maybe those of you attending class aren't the ones I need to be telling.)
- This week's Reflection will be up a bit later than usual; sorry about that. (Look for it after 4pm today!).

Matrices and Linear Maps

Earlier, we motivated the matrix-vector product by saying that it was like a function. Let's make that idea precise.

p. 172 **Definition.** Let $A \in M_{m,n}(\mathbb{R})$. The function $f_A : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{codomain}}$, called the *matrix map* (or *mapping*) corresponding to A, is the map defined by $f_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Example. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \end{bmatrix}$. $A \in M_{2,3}(\mathbb{R})$.

So $f_A : \mathbb{R}^3 \to \mathbb{R}^2$. $f_A\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} \in \mathbb{R}^2$.

$f_A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ | $f_A(\vec{e}_2) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ | $f_A(\vec{e}_3) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

p. 173 **Theorem** (3.2.1)**.**
Let $A = [A_1 \ \dots \ A_n] \in M_{m,n}(\mathbb{R})$, and $\vec{x} \in \mathbb{R}^n$. $\left(\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)$.
Then $f_A(\vec{x}) = x_1 A_1 + \dots + x_n A_n = x_1 f_A(\vec{e}_1) + \dots + x_n f_A(\vec{e}_n)$.

Proof. Since $f_A(\vec{e}_i) = A\vec{e}_i = A_i$, we have:

$f_A(\vec{x}) = A\vec{x} = x_1 A_1 + \dots + x_n A_n = x_1 f_A(\vec{e}_1) + \dots + x_n f_A(\vec{e}_n)$.

∎

Example continued.

$f_A\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = 2 f_A(\vec{e}_1) + 1 f_A(\vec{e}_2) + 2 f_A(\vec{e}_3)$
$= 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$.

Previously, we saw that $A(s\vec{v} + t\vec{w}) = sA\vec{v} + tA\vec{w}$. In a sense, the map f_A is "preserving the structure" of linear combinations. This idea is also important!

p.174 **Definition.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say that L is a *linear* map (or transformation) when for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $s, t \in \mathbb{R}$, we have $L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$.

* Remind of subspaces! *

Example. • Every matrix map (f_A) is linear (as noted previously).

• Let $L_1 : \mathbb{R}^n \to \mathbb{R}^n$ be given by $L_1(\vec{x}) = \vec{x}$. Linear?

$L_1(s\vec{x} + t\vec{y}) = s\vec{x} + t\vec{y}$ (by def'n of L_1) $= s(L_1(\vec{x})) + t(L_1(\vec{y}))$. ✓ Yes, L_1 is linear.

• Let $L_2 : \mathbb{R}^n \to \mathbb{R}^n$, $L_2(\vec{x}) = \vec{0}$. Linear?

$L_2(s\vec{x} + t\vec{y}) = \vec{0}$ | $sL_2(\vec{x}) + tL_2(\vec{y}) = s\cdot\vec{0} + t\cdot\vec{0} = \vec{0} + \vec{0} = \vec{0} = L_2(s\vec{x} + t\vec{y})$. ✓ Yes, L_2 is linear.

• Let $L_3 : \mathbb{R}^1 \to \mathbb{R}^2$, $L_3(x) = x\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Linear?

Line in $\mathbb{R}^2$ not through origin.

Nope! Why not? $L_3((1)(1) + (1)(1)) = L_3(2) = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$. Not linear!

$(1)L_3(1) + (1)L_3(1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix} \neq \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

• Let $L_4 : \mathbb{R}^2 \to \mathbb{R}^2$, $L_4\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_1 - x_2 \end{bmatrix}$. Linear?

$L_4\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2\cdot 2 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. $2L_4\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1\cdot 1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

→ Not linear!

p.180 **Definition.** The map id : $\mathbb{R}^n \to \mathbb{R}^n$ (or Id, or Id_n, or id_n) is called the *identity map* on $\mathbb{R}^n$.

$L: \mathbb{R}^3 \to \mathbb{R}^5$

Example. Let L be given by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_3 \\ x_1 + 2x_2 + 3x_3 \\ x_2 \\ 0 \\ 42x_3 \end{bmatrix}$.

Is L Linear?

Option 1: Use the definition.

Option 2: Show that L is actually a matrix map! (Know to be linear).

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_3 \\ x_1 + 2x_2 + 3x_3 \\ x_2 \\ 0 \\ 42x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 3x_3 \\ 0 \\ 0 \\ 42x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 42 \end{bmatrix}$$

$$= \underbrace{\underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 42 \end{bmatrix}}_{5\times 3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times 1}}_{\text{matrix map}}$$

$[L]_{S_2 \leftarrow S_3}$ ($[L]$) $= A$

Since $L = f_A$, we see that L is linear.

"Every linear map is a matrix map".

p. 176 **Theorem** (3.2.3). Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.
Let $S_n = \{\vec{e}_1, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$ (standard basis), for each n.
Let $[L]_{S_m \leftarrow S_n} = [L(\vec{e}_1) \ldots L(\vec{e}_n)] \in M_{m,n}(\mathbb{R})$.
Then for all $\vec{x} \in \mathbb{R}^n$, $L(\vec{x}) = [L]_{S_m \leftarrow S_n} \vec{x}$.

p. 176 **Definition.** The matrix $[L]_{S \leftarrow S}$ is called the *standard matrix* for L (or, the *matrix of L from $\underset{(n)}{S}$ to $\underset{(m)}{S}$*). (sometimes we'll just write $[L]$...)

Example. Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = \text{proj}_{\vec{v}}(\vec{x})$. Show that L is linear and find $[L]_{S\leftarrow S}$.

Solution. First, Let $\vec{x}, \vec{y} \in \mathbb{R}^2$, $s,t \in \mathbb{R}$. Then we have:

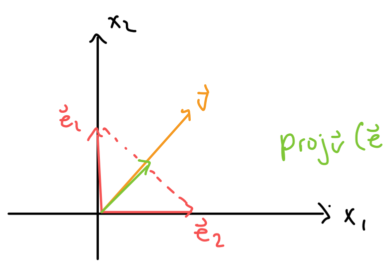
$$\text{proj}_{\vec{v}}(s\vec{x}+t\vec{y}) = \frac{(s\vec{x}+t\vec{y})\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{(s(\vec{x}\cdot\vec{v})+t(\vec{y}\cdot\vec{v}))}{\|\vec{v}\|^2}\vec{v} = s\left(\frac{(\vec{x}\cdot\vec{v})}{\|\vec{v}\|^2}\vec{v}\right)+t\left(\frac{(\vec{y}\cdot\vec{v})}{\|\vec{v}\|^2}\vec{v}\right)$$

$$= s\,\text{proj}_{\vec{v}}(\vec{x}) + t\,\text{proj}_{\vec{v}}(\vec{y}). \quad \checkmark$$

To compute $[L]$, we need to compute $L(\vec{e}_1)$ and $L(\vec{e}_2)$:

$$\text{proj}_{\vec{v}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1^2+1^2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Big| \quad \text{proj}_{\vec{v}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{1^2+1^2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So $[L] = \left[L(\vec{e}_1) \;\vdots\; L(\vec{e}_2) \right] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.



$\text{proj}_{\vec{v}}(\vec{e}_1) = \text{proj}_{\vec{v}}(\vec{e}_2)$.

When we motivated matrix multiplication, we said that because matrix-vector multiplication was like a function, then there should be some sort of composition aspect for matrices. We have the same thing for linear maps! $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m \xrightarrow{M} \mathbb{R}^p$

p.179 **Definition.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear maps. The *composition* $M \circ L$ of linear maps is the map from $\mathbb{R}^n$ to $\mathbb{R}^p$ defined by

$$M \circ L\,(\vec{x}) = M(L(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^n.$$

$\boxed{\texttt{\\circ}}$ in LaTeX.

Note. Because L and M are linear, $M \circ L$ is also linear.

Example. Let $L : \mathbb{R}^3 \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$L(\vec{x}) = x_1 + x_2 - 3x_3, \qquad M(y) = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Both are linear!

Find $M \circ L$ and compute $[M \circ L]$. Let $\vec{x} \in \mathbb{R}^3$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then:

$M \circ L(\vec{x}) = M(L(\vec{x})) = M(x_1 + x_2 - 3x_3) = (x_1 + x_2 - 3x_3)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

To find $[M \circ L]$:

1) $M \circ L(\vec{x}) = (x_1 + x_2 - 3x_3)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -6 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $\Rightarrow$

$[M \circ L] = \begin{bmatrix} 2 & 2 & -6 \\ 1 & 1 & -3 \end{bmatrix}$.

or:

2) Compute each $M \circ L(\vec{e}_i)$! $M \circ L(\vec{e}_1) = (1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = M \circ L(\vec{e}_2)$

$M \circ L(\vec{e}_3) = -3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \end{bmatrix} \longrightarrow [M \circ L] = \begin{bmatrix} 2 & 2 & -6 \\ 1 & 1 & -3 \end{bmatrix}$.

p.180 **Theorem** (3.2.5). Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear maps.

Then $[M \circ L]_{S_p \leftarrow S_n} = [M]_{S_p \leftarrow S_m}[L]_{S_m \leftarrow S_n}$.

Proof. Use defn's of matrix multiplication and standard matrix!

Compute $M \circ L(\vec{e}_i) = M(L(\vec{e}_i)) = M(\underbrace{[L]_i}_{i^{th} \text{ column of } [L]!}) = [M][L]_i$.

$\Rightarrow [M \circ L] = [[M][L]_1, \ldots, [M][L]_n] = [M][L]$, by definition ∎