May 31 (Lecture 7)

Overview: We'll briefly talk about bases and how the rank of a matrix relates to spanning sets and linear independence. Then, it's on to matrices and "matrix algebra"!

Learning Goals:

- Define and check for subspaces and bases for subspaces in $\mathbb{R}^n$.
- Correctly perform basic computations with matrices.

As you're getting settled:

- Homework 3 due Tuesday, 11:30 pm Pacific.
- Please reach out and ask for help instead of violating academic integrity.
- HW3 Q3 - Hints posted on the forum!

Chapter 3

p.147 Operations on Matrices

Definition. The set of all $m \times n$ matrices (with real entries) is denoted $M_{m,n}(\mathbb{R})$ (or $M_{m\times n}(\mathbb{R})$). The entries of a matrix A are denoted A_{ij}. Two matrices A and B in $M_{m,n}(\mathbb{R})$ are *equal* when
(or $A_{i,j}$) $A_{ij} = B_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

p.148 **Definition.** An $m \times n$ matrix A is *square* when $m=n$.
(and "rectangular" if $m \neq n$) (Joseph prefers "non-squares").

A square matrix A is *upper* or *lower triangular* when $A_{ij}=0$ for all $i>j$ (upper), or $A_{ij}=0$ for all $i<j$ (lower).

A square matrix A is *diagonal* when $A_{ij}=0$ for $i \neq j$

Example. $\begin{bmatrix} 1 & \boxed{2} \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & \boxed{1} \\ 3 & 4 \end{bmatrix}$.

$\begin{bmatrix} 0 & -1 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is upper triangular

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 2 & 2 & 4 \end{bmatrix}$ is lower triangular

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is diagonal

Notation: diag$(2,5,6)$

p.149 **Definition.** The *sum* of two matrices $A, B \in M_{m,n}(\mathbb{R})$, denoted $A+B$, is given by $(A+B)_{ij} = A_{ij} + B_{ij}$

The *scalar multiplication* of $A \in M_{m,n}(\mathbb{R})$ by a real number c, denoted cA, is given by $(cA)_{ij} = cA_{ij}$,

Both operations are done entrywise. (Entry by Entry).

- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ -4 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ -4 & 0 & 2 \end{bmatrix}.$

Example.

- $-3\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 3 & 0 \\ -3 & -12 \end{bmatrix}.$

#Same as
Vector theorem
1.4.1

Theorem (3.1.1)**.** *For all* $A, B, C \in M_{m,n}(\mathbb{R})$ *and* $s, t \in \mathbb{R}$*, we have:*

1. $A + B \in M_{m,n}(\mathbb{R})$
2. $A + B = B + A$
3. $(A + B) + C = A + (B + C)$
4. *There is a matrix* $O_{m,n}$ *(the* zero *matrix) such that* $A + O_{m,n} = A$
5. *For each* A *there is* $-A \in M_{m,n}(\mathbb{R})$ *such that* $A + (-A) = O_{m,n}$
6. $tA \in M_{m,n}(\mathbb{R})$
7. $s(tA) = (st)A$
8. $(s + t)A = sA + tA$
9. $t(A + B) = tA + tB$
10. $1A = A$

Note. These properties should look familiar! (Theorem 1.4.1) Matrices have many of the same properties as vectors; sometimes we say they have the same "structure". We can also do *linear combinations* of matrices, just like for "column" vectors.

p.152 **Definition.** Let $A \in M_{m,n}(\mathbb{R})$. The *transpose* of A is the $n \times m$ matrix A^T, with entries given by $(A^T)_{ij} = A_{ji}$, for $1 \le i \le n$, $i \le j \le m$.

Example.

$$\underset{2 \times 3}{\begin{bmatrix} -1 & 0 & 4 \\ 2 & 3 & 5 \end{bmatrix}^T} = \underset{3 \times 2}{\begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 4 & 5 \end{bmatrix}}.$$

p.153 **Theorem** (3.1.2). *For $A, B \in M_{m,n}(\mathbb{R})$ and $s \in \mathbb{R}$, we have:*

- $(A^T)^T = A$.
- $(A+B)^T = A^T + B^T$.
- $(sA)^T = sA^T$.

Proof. For ↗, note that both matrices are $n \times m$.

For $1 \le i \le n$, $1 \le j \le m$, we have:

$$((sA)^T)_{ij} = (sA)_{ji} = sA_{ji} = s(A^T)_{ij} = (sA^T)_{ij}.$$

all by the definitions!

This EQ holds for all entries, this $(sA)^T = sA^T$. ∎

Note.

- What is $M_{1,1}(\mathbb{R})$? "Essentially" just $\mathbb{R}$! $[c]$ "=" c.
- What is $M_{n,1}(\mathbb{R})$? "Essentially" $\mathbb{R}^n$! $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow$ many of the upcoming theorems have similarlies to theorems where seen before, or apply equally for vectors/scalars!

Matrix Multiplication

Motivation. Recall that given an SLE with coefficient matrix A and vector of constants $\vec{b}$, we can write the system of equations as an equality of vectors:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \longrightarrow x_1\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix} + x_2\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

<u>Idea</u>: Define "$A\vec{x}$" so that ⤴ becomes $A\vec{x}=\vec{b}$.

p.157 **Definition.** Let A be an $m \times n$ matrix with columns $A_1, \ldots, A_n$, and let $\vec{v} \in \overset{\mathbb{R}}{\cancel{\mathbb{C}}}{}^n$ be a vector of size n. Then the *matrix-vector product* of A with $\vec{v}$, denoted $A\vec{v}$, is defined to be

$$A\vec{v} = \left[A_1, \ldots, A_n\right]\begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix} := v_1A_1 + \ldots + v_nA_n$$

LC of columns of A by entries of $\vec{v}$.

$$= \begin{bmatrix}1&2&5\\2&4&6\end{bmatrix}$$

Example. Let $A = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$, $B = A^T$, $\vec{v} = \begin{bmatrix}-1\\1\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}2\\0\\1\end{bmatrix}$.

Compute each of $A\vec{v}$, $A\vec{w}$, $B\vec{v}$, and $B\vec{w}$, or explain why the product doesn't make sense.

Solution. $$A\vec{v} = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = (-1)\begin{bmatrix}1\\3\\5\end{bmatrix} + (1)\begin{bmatrix}2\\4\\6\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

2 columns 2 entries

‖ ☺

$A\vec{w}$ is <u>not</u> defined.

2 columns 3 entries

‖ ☹

$B\vec{v}$? is <u>not</u> defined.

3 cols 2 entries

‖ ☹

Example continued.

$$\underset{\substack{\text{3 cols}\\ \shortparallel\\ U}}{B}\underset{\text{3entries}}{\vec{w}} = \begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}\begin{bmatrix}2\\0\\1\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix} + \cancelto{0}{0\begin{bmatrix}3\\4\end{bmatrix}} + (1)\begin{bmatrix}5\\6\end{bmatrix}$$

$$= \begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}7\\10\end{bmatrix}$$

Note. In $B\vec{w}$, look at the result entry-by-entry:

$$(B\vec{w})_1 = 7 = (1)(2) + (3)(0) + (5)(1) = \begin{bmatrix}1\\3\\5\end{bmatrix}\circ\begin{bmatrix}2\\0\\1\end{bmatrix}.$$

$$(B\vec{w})_2 = 10 = \cdots = \begin{bmatrix}2\\4\\6\end{bmatrix}\circ\begin{bmatrix}2\\0\\1\end{bmatrix}.$$

In general: If $A = \begin{bmatrix}R_1\\ \vdots\\ R_m\end{bmatrix} \in M_{m\times n}(\mathbb{R})$ and $\vec{v} \in \mathbb{R}^n$, then $A\vec{v} = \begin{bmatrix}R_1^T\cdot\vec{v}\\ \vdots\\ R_m^T\cdot\vec{v}\end{bmatrix}.$

Example.

$$\underset{\nearrow\\ A}{\begin{bmatrix}1&0&0\\2&1&1\\-1&3&3\end{bmatrix}}\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0+0+0\\2\cdot0+1\cdot1+1\cdot1\\-1\cdot0+3\cdot1+3\cdot-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow A\vec{v} = \vec{0}$$

$\rightarrow \vec{v}$ is a solution to the homogeneous SLE w/ coeff. matrix A.

Note. What happens if we try to compute $\underset{m\times n}{A}(\underline{s\vec{v}} + \underset{\in\mathbb{R}^n}{\underline{t\vec{w}}})$?

$$(A(s\vec{v}+t\vec{w}))_i = R_i^T\circ(s\vec{v}+t\vec{w}) = s(R_i^T\cdot\vec{v}) + t(R_i^T\cdot\vec{w})$$

$$= s(A\vec{v})_i + t(A\vec{w})_i$$

for $\underline{\text{all}}$ i).

$$\implies A(s\vec{v}+t\vec{w}) = sA\vec{v} + tA\vec{w}$$

Note. What is $\underset{m\times n}{A}\underset{\in\mathbb{R}^n}{\vec{e}_i}$? $A = [A_1 \cdots A_n]$, $\vec{e}_i = \begin{bmatrix}0\\0\\ \vdots\\1\\ \vdots\\0\end{bmatrix} \leftarrow$ ith entry.

$$A\vec{e}_i = 0A_1 + \ldots + 0A_{i-1} + A_i + 0A_{i+1} + \ldots + 0A_n = A_i$$

Motivation for Matrix Multiplication. Is there a way to compute "AB"? What *should* it be?

- "$A\vec{v}$" looks a lot like a function: Input: $\vec{v} \in \mathbb{R}^n$
  Output: $A\vec{v} \in \mathbb{R}^m$
  ($A \in M_{m,n}(\mathbb{R})$)
- functions can be composed: "$f \circ g$ = apply g, then apply f"
- "$AB\vec{v}$" should be the same regardless of doing $B\vec{v}$, then $A(B\vec{v})$, or $(AB)\vec{v}$.
  (associative property).

So: $\underset{\text{"should"}}{A B\vec{v}} = A(v_1B_1 + \dots v_nB_n) \underset{LC}{=} v_1 \underbrace{AB_1}_{} + \dots + v_n \underbrace{AB_n}_{}$.

$$= \underbrace{[AB_1 \cdots AB_n]}_{\text{good idea!}}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

p.161 **Definition.** Let A be an $m \times n$ matrix. Let $B = [B_1 \cdots B_p]$ be an $n \times p$ matrix. Then the *matrix product* of A with B, denoted AB, is the $m \times p$ matrix $AB = [AB_1 \cdots AB_p]$.

Note. To compute AB, we can use either matrix-vector product computation!

Example.

$$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} = \begin{cases} \left[(1)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 0 \end{bmatrix} \;\vdots\; (0)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 0 \end{bmatrix} \right] \\ \begin{bmatrix} (1)(1)+(2)(1) & \vdots & (1)(0)+(2)(1) \\ (3)(1)+(0)(1) & \vdots & (3)(0)+(0)(1) \end{bmatrix} \end{cases} = \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}$$

☺

Note: "$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$" is the same as $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$!

Example. Let $A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}}$ and $B = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$. Compute AB and BA, if possible.

$$\underset{\underset{2\times 3 \quad 3\times 3}{}}{AB} = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 1 & 0 & 1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 1+3 & 2 & 1+3 \\ 1 & -1 & 1 \end{bmatrix}} = \begin{bmatrix} 4 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

(lots of heuristics or computational shortcuts).

$\underset{\underset{3\times 3 \quad 2\times 3}{}}{BA}$ is not defined (# columns of $B \neq$ # rows of A).

Note. $AB \neq BA$. They might not even be defined, let alone the same size.

p.165 Definition: The identity matrix of size n is $I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$ $= \operatorname{diag}(1, \ldots, 1)$. $(n\times n)$

p.165 **Theorem** (3.1.6). *For $A \in M_{m,n}(\mathbb{R})$, we have $I_m A = A = AI_n$.*

Proof.
- $(I_m A)_{ij} = \underset{\in \mathbb{R}^m}{\vec{e}_i} \cdot A_j = A_{ij}$ for all i, j, so $I_m A = A$!
- $AI_n = [A\vec{e}_1 \ldots A\vec{e}_n] = [A_1 \ldots A_n] + A$!

$$\downarrow$$

$$R_i \begin{bmatrix} 0 \ldots 010 \ldots 0 \end{bmatrix} \begin{bmatrix} \cdots A_j \cdots \end{bmatrix}$$

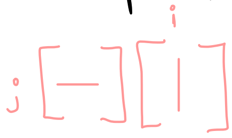
▨

We used both multiplication expressions! Very cool.

p.163 **Theorem** (3.1.4). *For matrices $A \in M_{m,n}(\mathbb{R})$, $B, C \in M_{n,p}(\mathbb{R})$, $D \in M_{p,r}(\mathbb{R})$, and $s \in \mathbb{R}$, we have:*

- $A(B+C) = AB + AC$
- $(B+C)D = BD + CD$
- $s(AB) = (sA)B + A(sB)$
- $A(BD) = (AB)D$
- Midterm 1! $(AB)^T = B^T A^T$.

Proof. $(AB)^T = B^T A^T$. AB is $m \times p$, so $(AB)^T$ is $p \times m$, so is $B^T A^T$.



For $1 \le i \le p$, $1 \le j \le m$, we have:

$$((AB)^T)_{ij} = (AB)_{ji} = (\text{Row}_j \text{ of } A)^T \cdot (\text{Col}_i \text{ of } B)$$
$$= (\text{Col}_j \text{ of } A^T) \cdot (\text{Row}_i \text{ of } B^T)^T$$
$$= (B^T A^T)_{ij} \quad \Rightarrow (AB)^T = B^T A^T$$

def'n

∎

p.164 **Theorem** (3.1.5).
If $A, B \in M_{m,n}(\mathbb{R})$ and $A\vec{x} = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ (or just for $\vec{e}_1, \dots, \vec{e}_n$), then $A = B$.

Proof. For each $i = 1, \dots, n$, we have $\overset{\text{ith col of } A}{\downarrow} \; \overset{\text{ith col of } B}{\downarrow}$ $A_i = A\vec{e}_i = B\vec{e}_i = B_i$.
Each of the columns are the same, and so $A = B$.

$$\left(\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ entry}\right).$$

∎