May 27 (Lecture 6)

Overview: After working a bit with homogeneous systems, we'll use SLEs to discuss the concepts of spanning sets and linear independence.

Learning Goals:

- Define and check for "spanning" and "linearly independent" sets in the context of $\mathbb{R}^n$.
- Define and check for subspaces and bases for subspaces in $\mathbb{R}^n$.

As you're getting settled:

- Homework 3 is out! Due Tuesday, 11:30 pm Pacific.
- Reflection available after class, due Friday night, 11:30 pm!
  - By the way, I <u>do</u> read the Reflections (anonymously, mostly).

p.51 **Definition.** A non-empty subset S of $\mathbb{R}^n$ is called a *subspace* (of $\mathbb{R}^n$) when for all $\vec{v}, \vec{w} \in S$ and $s, t, \in \mathbb{R}$, we have $s\vec{v} + t\vec{w} \in S$.
i.e. S is closed under linear combinations.

Note. Would also say: $\Big\{$ • S is closed under vector addition. (set $= 1$). #1
• S is closed under scalar multiplication ($\vec{w} = \vec{0}$). #6
theorem 1.4.1

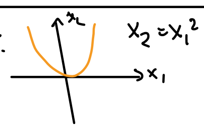
Example.

- $S = \mathbb{R}^n$ is a subspace! (Thm 1.4.1) → trivial
- $S = \{\vec{0}\} \subseteq \mathbb{R}^n$ is also a subspace: $s\vec{0} + t\vec{0} = \vec{0} + \vec{0} = \vec{0}$ → subspaces!
- A line through $\vec{0}$, $L = \{t\vec{v} : t \in \mathbb{R}\} \subseteq \mathbb{R}^n$: $s(t_1\vec{v}) + r(t_2\vec{v}) = (st_1 + rt_2)\vec{v} \in L$. ($\vec{v} \neq \vec{0}$).
- A plane through $\vec{0}$, $P = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; 3x_1 + 2x_2 - 4x_1 = 0 \right\} \subseteq \mathbb{R}^3$. Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in P$, $s, t \in \mathbb{R}$

then $s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + t\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ sx_3 + ty_3 \end{bmatrix}$, check if the equation holds!

$$\begin{aligned} 3(sx_1 + ty_1) &= 2(sx_2 + ty_2) - 4(sx_3 + ty_3) \\ &= s(3x_1 + 2x_2 - 4x_3) + t(3y_1 + 2y_2 - 4y_3) \\ &= s(0) + t(0) = 0. \end{aligned}$$

- Quadratic in $\mathbb{R}^2$:



$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in Q$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and $2 \neq 2^2 = 4$,
so $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin Q$. So Q is not a subspace of $\mathbb{R}^2$.

$Q = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 = x_1^2 \right\}$.

* (more eg's in filled in notes)!

p.53 **Theorem** (1.4.2). If S is a subset of $\mathbb{R}^n$, the span S is a subspace of $\mathbb{R}^n$.

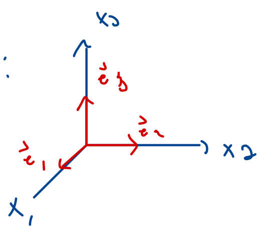
[eg. $4(3\vec{v} + 5\vec{w}) + 2(-\vec{v} + \vec{w}) = 10\vec{v} + 22\vec{w}$.
a LC of $\vec{v}$ and $\vec{w}$.]

Note: plural of basis = "bases"

p.55 **Definition.** Let W be a subspace of $\mathbb{R}^n$. A subset B of W is a *basis* for W when B is a linearly independant spanning set for W.
Bases help us describe subspaces using LC's w/out unnecessary repitition.

Example.
for $i = 1, \cdots, n$ let $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \boxed{1} \\ \vdots \\ 0 \end{bmatrix}$ ← ith entry. The $B = \{\vec{e}_1, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$ is the standard basis for $\mathbb{R}^n$.

In $\mathbb{R}^3$:



to check if B is a basis for $\mathbb{R}^n$, we check if B is LI, and if $\operatorname{span} B = \mathbb{R}^n$.

Spanning: for $\vec{x} \in \mathbb{R}^n$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$= x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n \in \operatorname{span} B$ ✓

LI: If $c_1 \vec{e}_1 + \ldots + c_n \vec{e}_n = \vec{0}$, the SLE is represented by

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

RREF already! → zero vector
Homogeneous system

→ The only solution is $c_1 = c_2 = \cdots = c_n = 0$.

Thus B is linearly independent. ✓

↘ B is a Basis for $\mathbb{R}^n$.

From a previous example the solution set to a homogeneous SLE was...

$$\left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_3, x_5 \in \mathbb{R} \right\} = \operatorname{span} \underbrace{\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{B_1}.$$

B_1 spans the solution set, and its easy to see that B_1 is LI. So B_1 is a basis for the solution set. (which is a subspace).

LI = "linearly independant"

Theorem. *Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$. Let* nxk → *A be the coefficient matrix of the homogeneous system $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ (its columns are the vectors $\vec{v}_1, \ldots, \vec{v}_k$).*

(2.3.1) S spans $\mathbb{R}^n$ if and only if $\text{rank}(A) = n$.

(2.3.2) If S spans $\mathbb{R}^n$, then $k \geq n$. (spanning sets have to be at least a certain size).

(2.3.3) S is LI if and only if $\text{rank}(A) = k$.

(2.3.4) If S is LI, then $k \leq n$. (LI sets can only be so big).

If $k = n$, then we also have:

(2.3.5) S is a basis for $\mathbb{R}^n$ if and only if $\text{rank}(A) = n$.

(2.3.6) S spans $\mathbb{R}^n$ if and only if S is LI.

Example.

- $S_1 = \{\vec{v}_1, \ldots, \vec{v}_5\} \subseteq \mathbb{R}^3$. By thm 2.3.4, S_1 cannot be LI. ($5 \nleq 3$).
- $S_2 = \{\vec{v}_1, \vec{v}_2\} \subseteq \mathbb{R}^4$. By thm. 2.3.2, S_2 cannot span $\mathbb{R}^4$.
- $S_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. $k = n = 3$, so S_3 spans $\mathbb{R}^3$ if and only if S_3 is LI.
- $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & 1 \\ 2 & -4 & 0 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}$. $\text{rank}(A) = 2$, so by thm. 2.3.5, S_3 is not a basis for $\mathbb{R}^3$ (not spanning nor LI).

Note. By row-reducing a matrix, we can:

- Solve SLE's
- check for membership in a span.
- Check LI
- check spanning set
- check for basis.

An application. The textbook has a variety of discipline-specific examples in section 2.4, none of which I would do justice in class. So, here's a mathematics application.

Suppose (x, y) data is known to fit a quadratic equation of the form $y = f(x)$, with known data points $(-1, 1), (1, 1), (2, -2)$. Find the explicit equation.

We know $y = f(x) = a + bx + cx^2$, for some $a, b, c \in \mathbb{R}$.

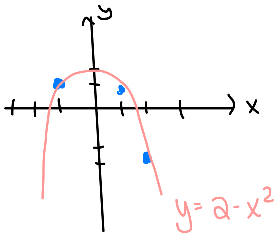
Substitute data points into ↑:

$$\begin{aligned} 1 &= a - b + c \\ 1 &= a + b + c \\ -2 &= a - 2b + 4c \end{aligned} \xrightarrow[a,b,c!]{\text{SLE in}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -2 & 4 & -2 \end{array}\right] \xrightarrow[\text{ops}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Our solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Thus, the quadratic equation is

$$\boxed{y = 2 - x^2.}$$

Picture:



(Hey look, our solution does make sense!).

- Note how a quadratic isn't "linear" on its own, but by looking at the coefficients, we found a linear algebra problem! ☺