May 20 (Lecture 5)

Overview: We will define the reduced row echelon form (RREF) and the rank of a matrix and see how those two concepts can help us solve SLEs! We'll then use SLEs to discuss the concepts of spanning sets and linear independence.

Learning Goals:

- Precisely define "RREF" and determine whether or not a matrix is in RREF.
- Precisely define and compute the rank of a matrix.
- Define "span" and "linear independence" in the context of $\mathbb{R}^n$.

As you're getting settled:

- No class on May 24! (statutory holiday in Canada)
  Office hours to be held on Tuesday, 1:00-2:00 pm.
- Homework 2 is out! Due next Tuesday, 11:30 pm Pacific.
- Reflection available after class, due Friday night, 11:30 pm!
- Homework 1 Solutions Posted to Brightspace
- \rop right arrow command tex stacks.

Example continued.

These last few examples foreshadow this theorem: "System-Rank Thm."

p. 108 **Theorem** (2.1.3, 2.2.2). *Let $[A|\vec{b}]$ represent a system of m linear equations in n variables.* $n = \#$ columns of A.

1. The system is consistent if and only if $\text{rank}(A) = \text{rank}[A|\vec{b}]$.
   (or: ..... is inconsistent if and only if the RREF of $[A|\vec{b}]$ has a row of $[0 \dots 0 | 1]$!).

2. If the system is consistent, then the number of free variables (parameters) is $n - \text{rank}(A)$. (unique solution $\Leftrightarrow n = \text{rank}(A)$).

3. We have $\text{rank}(A) = n$ if and only if the system represented by $[A|\vec{b}]$ is consistent for every $\vec{b} \in \mathbb{R}^n$.

Example. Suppose that the augmented matrix $[A|\vec{b}]$ for a system of linear equations has RREF equal to

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{0} & 0 & \boxed{0} & 0 \end{array}\right].$$

$m = 4$ "Rows or # of linear eq."

$n = 5$ ex. $(x_1, x_2, x_3, x_4, x_5)$

$\text{rank}[A|\vec{b}] = 3$

$\text{rank}(A) = 3$ also (same row ops will get A to RREF).

Is the system consistent? If so, how many parameters/free variables are there in the solution?

Solution. $\text{rank}(A) = \text{rank}[A|\vec{b}]$, so yes consistent!

\# of free variables: $n - \text{rank}(A) = 5 - 3 = \boxed{2}$

x_3, x_5

$x_1 + x_2 + 3x_3 = 0$

p.109 **Definition.** A linear equation is *homogeneous* when the constant term is 0.
A system of linear equations is *homogeneous* when each EQ in the system is homogeneous. ($\vec{b} = \vec{0}$).
$[A \mid \vec{0}]$

Example. Consider the SLEs represented by augmented matrices

→ free variables

$$M_1 = \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad M_2 = \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 2 & 1 \\ 0 & \boxed{1} & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A A

Is there a relationship between the solution sets for these systems?

- M_1 represents a homogeneous system w/ coefficient matrix A;
- M_2 represents a non-homogeneous system w/ coefficient matrix A; and vector of constants $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}$.

→ Both M_1 and M_2 are in RREF!

M_1: $x_1 - x_3 + 2x_5 = 0$
$x_2 + x_3 - 3x_5 = 0$
$x_4 = 0$
$0 = 0$

$\longrightarrow$ solution vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_5 \\ -x_3 + 3x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Solution set $= \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ x_3, x_5 \in \mathbb{R} \right\}$

M_2: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \ldots = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}}.$

Solutions to a non-homogeneous system are equal to a "particular" solution + the "homogenous" solution.

p.61 **Definition.** For a homogeneous SLE, the zero vector is the *trivial solution* to the system.

Spanning and Linear Independence in $\mathbb{R}^n$

p.52 **Definition.** Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. The *span* of S, denoted span S or span(S), is the set of all linear combinations of vectors in S:

$$\operatorname{span} S = \{ c_1\vec{v}_1 + \ldots + c_k\vec{v}_k : c_1, \ldots c_k \in \mathbb{R} \}$$

If $W = \operatorname{span} S$, then S is a <u>spanning</u> set for W, or S <u>spans</u> W.

Example. Is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in the span of the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$?

Solution. Check if $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

<u>Rephrased:</u>
$$c_1 - c_2 = 1$$
$$2c_1 + c_2 = 0$$
$$c_2 = -1$$

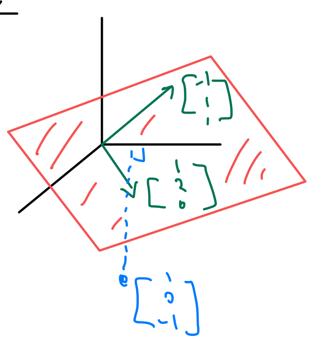
Easily solvable, but we could use a matrix!

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow[R_1 - R_3]{R_2 - 3R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + R_3} \underbrace{\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{\text{RREF!}}.$$

We have a row representing $0 = 1$, so the SLE is <u>inconsistent.</u>

Thus, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is <u>not</u> in span S.

<u>Picture</u>



Example. What is the span of $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$?

$\operatorname{span} S = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}$

From the last eg. we know the span contains a plane. Does it contain <u>more</u> vectors, thanks to $\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$?

$\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\text{ops}]{\text{Row}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

Solution to SLE ↑: $c_1 = 1$, $c_2 = 2$, so $1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$! so $\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ does <u>not</u> change the span i.e. $\operatorname{span} S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

p.53 **Definition.** Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. The set S is *linearly dependent* when there exists real numbers, $c_1, \ldots, c_k$ <u>not all zero</u> such that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$.

S is linearly independent when S is not linearly dependent!

or: the only solution to $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$ is $c_1 = \cdots = c_k = 0$. "the trivial solution".

Example continued. From $1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

$\longrightarrow \underbrace{1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + -1 \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}}_{\text{"linear dependence relation on } S\text{."}} = \vec{0}$, so S is linearly dependent.

Example. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \right\}$. Is S linearly independent? If not, find a linearly independent subset S' of S such that $\operatorname{span} S' = \operatorname{span} S$.

Start w/ $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c_4 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

SLE in c_1, c_2, c_3, c_4! Let's use a matrix to solve the SLE:

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & -1 & 1 & -3 \\ 1 & 5 & 4 & 3 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 3/2 & -2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

RREF!

(No need to include the $\vec{b}$ as it contains all zeros.)

Read off the solution:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3/2\, c_3 + 2c_4 \\ 1/2\, c_3 - c_4 \\ c_3 \\ c_4 \end{bmatrix} = c_3 \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad c_3, c_4 \in \mathbb{R}$$

There are non-trivial solutions to the vector EQ;

For example, choose $c_3 = 1 = c_4$. $c_1 = -3/2 + 2 = 1/2$, $c_2 = 1/2 - 1 = 3/2$

Thus, $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

- $c_3 = 1, c_4 = 0$: $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ is a LC of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.
- $c_3 = 0, c_4 = 1$: $\begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$ is a LC of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

So if $S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \right\}$, then $\operatorname{span} S' = \operatorname{span} S$.

<u>And</u> S' is LI, by our above computation!