May 17 (Lecture 4)

Overview: Today we will focus on systems of linear equations and how to solve them, which will be an important tool throughout the rest of this course!

Learning Goals:

- Solve systems of linear equations.
- Precisely define "RREF" and determine whether or not a matrix is in RREF.

As you're getting settled:

- Homework 1 due tomorrow night, 11:30 pm!
- Homework 2 to come out tomorrow morning, based on the material from today/last Thursday.

p. 90 **Theorem** (2.1.2). *If two augmented matrices are row-equivalent, then the associated systems of linear equations are equivalent.*

p. 92, 104 **Definition.** Let A be a matrix. We say that A is in *row echelon form* (or *REF*) when:

$\ast \begin{array}{c} \begin{bmatrix} 1 & \cdots & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} i \\ j \end{array}$

1. All rows of 0's (if there are any) are at the bottom of the matrix.

"top-left to bottom-right diagonal pattern"

2. If row i is above row j, then the left most non-zero entry (Leading entry) of row i is to the left of the leading entry in row j. $\ast$

We say that A is in *reduced row echelon form* (or *RREF*) when it is in REF and:

3. All leading entries are <u>ones</u> (leading ones)

4. All entries above (and below) a leading one are zero.

If B is a matrix in (R)REF and is row-equivalent to A, then we say that B is an (R)REF for A.

Example. Determine whether or not the following matrices are in REF or RREF:

Rules #3 → ✓ ✓ $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ← #2 ← #1

REF

Rules 1 + 2 are satisfied.
⇒ in REF.

Rules 3 and 4 are also satisfied ⇒ in RREF.

$\begin{bmatrix} 0 & 2 & 1 & 4 \\ 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Rule 2 is not satisfied.
⇒ Not in REF.

⇒ Not in RREF.

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ No Rows of Zero ← #1

← Rule #2 does not hold.

⇒ Not in REF.

⇒ Not in RREF.

Note. Sometimes leading entries are called *pivots* (as if you're "pivoting" the row operations around that entry).

Example. Find a matrix in RREF that is row-equivalent to the following matrix.

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution. Perform row operations to make the rules hold.

$$\begin{bmatrix} 1 & 3 & 4 \\ \cancel{2}^{\to 0} & 4 & 2 \\ \cancel{-1}^{\to 0} & -1 & 2 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

in RREF!

"the RREF"
(many ways to get to the RREF).

→ REF

Note. As per Theorem 2.2.2, p.105, while a matrix can have many REFs (REF is <u>not</u> unique), it can only have <u>one</u> RREF (RREF <u>is</u> unique)!

Example. Here are some matrices in RREF that are row equivalent to augmented matrices representing systems of linear equations. What are the solution sets for those systems?

leading ones

$$A = \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \atop \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 4 \\ 0 & \boxed{1} & 2 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right] \qquad B = \begin{array}{ccc} x_1 & x_2 & b \end{array} \atop \left[\begin{array}{cc|c} 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad C = \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \atop \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

Solution. Write down the SLE for the matrix in RREF!

A. $\begin{cases} x_1 + x_3 = 4 \\ x_2 + 2x_3 = -1 \\ x_4 = 5 \end{cases} \longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - x_3 \\ -1 - 2x_3 \\ x_3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

x_3 is not constrained, so it can be anything! "free variable" (parameter)

"or "t""

Solution set $= \left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} : x_3 \in \mathbb{R} \right\}$

B: $\begin{cases} x_2 = 2 \\ 0 = 0 \end{cases} \longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

x_1 is not constrained by the SLE! → "free".

Solution set: $\left\{ \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\text{point}} + \underbrace{x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1 \cdot \text{direction vector}} : x_1 \in \mathbb{R} \right\}$ ⟶ line!

C: $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \\ 0 = 1 \end{cases} \longrightarrow$ this SLE is <u>inconsistent</u>! ⟶ solution set is empty!

p.107 **Definition.** Let A be a matrix. The *rank* of A, denoted rank(A), is the number of leading ones in the RREF of A.

Example continued. What are the ranks of the matrices A, B, C from last page?

rank(A) = 3 | rank(B) = 1 | rank(C) = 4

Example. What is the rank of this matrix from before?

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution.

from before: $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} \boxed{1} & 0 & -5 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow \text{rank}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ -1 & -1 & 2 \end{bmatrix} = 2.$

(In RREF)

Example. Based on just the numerical answer to the last example, is the following system consistent?

augmented matrix for the SLE.

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 4 & 2 \\ -1 & -1 & 2 \end{array}\right]$$

from the RREF: yes!

From just the rank (= 2):

unfortunately, no!

$\rightarrow$ Here's an augmented matrix w/ rank 2 that yields an inconsistent system:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{"} 0 = 1 \text{"}$$

$$x_1 + 3x_2 = 4$$
$$2x_1 + 4x_2 = 2$$
$$-x_1 - x_2 = 2$$