May 13 (Lecture 3)

Overview: The dot product allows us to talk about more geometry, and in particular allows us to define *projections*. Then, we'll start talking about linear equations, which we've seen a little bit already!

Learning Goals:

- Use the dot product to compute lengths of and angles between vectors.
- Correctly define and use projections of vectors onto other vectors.
- Define and solve systems of linear equations.

As you're getting settled:

- Homework 1 is out! It's due "next" Tuesday (at **11:30 pm** sharp). (Practice submitting with Homework 0 and the Video Walkthrough!) MAY 18th!
- Yesterday was Women in Mathematics Day! (Maryam Mirzakhani's birthday, for reference.) Forum Post to come soon :)
- Reflection due Friday, 11:30pm! (Released after class today)
- Do you see anything in the Brightspace Calendar?
  Yes!

(Chapter 2)

Systems of Linear Equations

p.79 **Definition.** A *linear equation* in the variables $x_1, \dots, x_n$ is an equation of the form $a_1 x_1 + \dots + a_n x_n = b$

a_i = "coefficients", b = "constant term"

Note: $(a_1, \dots, a_n, b \in \mathbb{R})$

Example. Which of the following equations are linear in the variables x_1, x_2?

• 0 is a real number

- $\pi x_1 - e x_2 = 42$ YES!
- $x_1 = x_2^2 - 1$ NO! (Parabola) Not a line!
- $\sin^2(x_1) + \cos^2(x_2) = 1$ No!
- $x_1 + x_2 = 1$, where $x_1 = \sin^2(t_1)$ and $x_2 = \cos^2(t_2)$. Yes! (but not in t_1, t_2)*
- $0 = 0$ Yes! $(a_1 = a_2 = b = 0)$
- $\vec{n} \cdot \vec{x} = 1$, where $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Yes!

  $\hookrightarrow (n_1 x_1 + n_2 x_2 - 1)$

Note: $0 = 3$ is not a true equation! but it is linear.

(geometrically)

Note. A linear equation can represent...

- A point, in $\mathbb{R}$: $ax = b$ $(a \neq 0)$
- A line, in $\mathbb{R}^2$: $a_1x_1 + a_2x_2 = b$
- A plane, in $\mathbb{R}^3$: $a_1x_1 + a_2x_2 + a_3x_3 = b$
- A "hyperplane"! Higher Dimensional

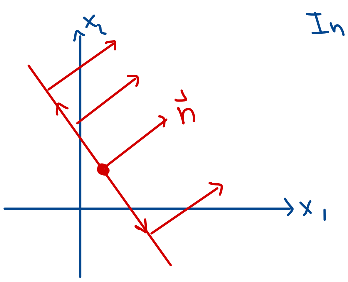
↳ more dimensions, but we can't draw them.

All of our equations look like $a_1x_1 + \ldots + a_nx_n = b$, so if $\vec{n} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then every linear equation looks like $\vec{n} \cdot \vec{x} = b$!

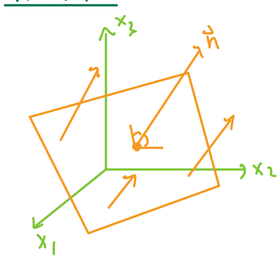
If $\vec{q}$ satisfies $\vec{n} \cdot \vec{q} = b$, then $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{q}$ can be rearranged to $\vec{n} \cdot (\vec{x} - \vec{q}) = 0$! Meaning, $\vec{n}$ is orthogonal to vectors in the hyperplane!

$\vec{n}$ is sometimes called the normal vector for the hyperplane.

In $\mathbb{R}^2$:



In $\mathbb{R}^3$:



In higher "dimensions", a hyperplane looks like "all vectors orthogonal to a given normal, translated a bit along the direction of the vector".

P. 80 (SLE)

Definition. A *system of linear equations* in variables $x_1, \ldots, x_n$ is a collection of m linear equations in those variables:

(or sometimes $a_{i,j}$: $a_{34,6}$)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_{\cancel{3}} + \cdots + a_{2n}x_n &= b_2 \\
&\;\;\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

2

In the notation $\boxed{a_{ij}}$: i-th equation, j-th variable coefficient.

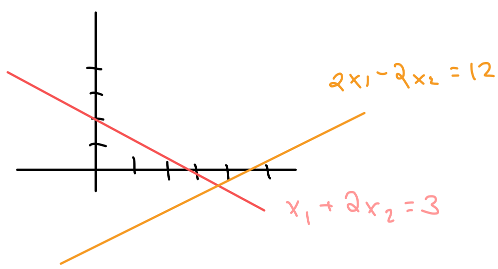
e.g. a_{23} → 2nd equation 3rd variable

A vector $\vec{x} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ is a *solution* of the system when all m equations are satisfied when $x_i = s_i$ for each i. The set of all solutions to a system is the *solution set*.

A system of linear equations is *consistent* when it has at least one solution; otherwise, it is *inconsistent*. i.e. No solutions

Example. $x_1 + 2x_2 = 3$

$2x_1 - 2x_2 = 12$



The solution set to this system is the intersection point of the lines!

Add EQ's together to get $3x_1 = 15$, or $x_1 = 5$

Sub $x_1 = 5$ back into $x_1 + 2x_2 = 3$ to get $5 + 2x_2 = 3$

(y) $2x_2 = -2$ or $x_2 = -1$

So, the solution set is $\left\{ \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\}$.

We can solve systems using three "equation" operations:

- Re-arrange EQ's
- Multiply an EQ by a non-zero scalar
- Replace one EQ with that EQ plus a scalar multiple of another EQ.

Example. Solve the following system of equations:

SLE in the variables x_1, x_2, x_3

$$
\begin{aligned}
x_1 - 2x_2 + 0x_3 &= -3 \\
0x_1 \quad -x_2 - x_3 &= -5 \\
x_1 - 2x_2 + x_3 &= 0
\end{aligned}
$$

Visualize the solution set.

Multiply EQ2 by -1:
(EQ1)
$x_2 + x_3 = 5$
(EQ3)

Replace EQ3 by EQ3 + (-1)(EQ1):
(EQ1)
$x_2 + x_3 = 5$
$x_3 = 3$

Let's see: $x_1 - 2x_2 + 0x_3 = -3$
$x_2 + x_3 = 5$
$x_3 = 3$

"Back-substitution"

$x_2 = 5 - x_3 = 5 - 3 = 2$

$x_1 = -3 + 2x_2 = -3 + 4$
$= 1$

So the solution set is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

☆ Visualizing:
- GeoGebra (3D)
- Grapher on macOS

Example. Find a vector in $\mathbb{R}^4$ orthogonal to both $\begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix}$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then...

$$0 = \vec{x} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix} = x_1 + 2x_2 + x_3 - 3x_4$$

$$0 = \vec{x} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix} = x_1 + 3x_2 + 0x_3 - 5x_4$$

Replace EQ2 with EQ3 + (-1) EQ1 : $x_1 + 2x_2 + x_3 - 3x_4 = 0$

"eliminate x_1 in EQ2"

$$(0x_1 +) 1x_2 - x_3 - 2x_4 = 0$$

Replace EQ1 with EQ1 + (-2) EQ2 : $x_1 (+ 0x_2) + 3x_3 + x_4 = 0$

"eliminate x_2 in EQ1" (new)

$$x_2 - 3x_3 - 2x_4 = 0$$

Solve for x_1, x_2 in terms of x_3, x_4 :

$$x_1 = -3x_3 - x_4$$
$$x_2 = x_3 + 2x_4$$
$$\begin{pmatrix} x_3 = x_3 \\ x_4 = x_4 \end{pmatrix}$$

Solutions are generally a vector EQ!

$$\vec{x} = \begin{bmatrix} -3x_3 - x_4 \\ x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

one possible vector is if $x_3 = 1 = x_4$:

$$\vec{x} = \begin{bmatrix} -4 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

p.85 **Definition.** If one can go from one system of linear equations to another via the three equation operations, then the two systems are *equivalent*. "This system are equivalent when... the two SLEs have the same solution set!"

Note. The process of solving systems of equations that we're using is called *(Gaussian) elimination*. If we get to a point halfway through and start substituting values back in, then sometimes we call it *back-substitution*.

Do we *really* need to keep track of the variable names?

p.88

Definition. Given a system of m linear equations in n unknowns, we can represent the system with a *matrix*:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right] \longleftarrow [A|\vec{b}].$$

Whole matrix with the line: "augmented matrix" associated to the SLE.

A – "coefficient matrix"

"vector of constants" – $\vec{b}$

Example. Write down the associated augmented matrix for this system of equations:

$$x_1 + x_3 = 3$$
$$-4x_1 + 3x_2 - x_3 = 6$$
$$2x_1 - x_2 + 3x_3 = 7$$

Solution: Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ -4 & 3 & -1 & 6 \\ 2 & -1 & 3 & 7 \end{array}\right]$$

coefficient matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -4 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Vector of constants:

$$\vec{b} = \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$$

p. 89

Definition. The "equation" operations have analogous *(elementary) row operations*:

- Swap two rows ($R_i \leftrightarrow R_j$)
- Multiply one row by a non-zero constant (aR_i) option 2: ($R_i \rightarrow$ _)
- Add a scalar multiple of R_j to R_i ($R_i + aR_j$)

We say that two matrices A and B are *row equivalent*, denoted $A \sim B$, when B can be obtained by performing a sequence of row operations to A. (or, vice versa! Row operations are reversible!)

"A tilde B"

Apply row operations

Example. Solve the following system of linear equations by "row-reducing" an appropriate matrix:

$$x_1 + x_3 = 3$$
$$-4x_1 + 3x_2 - x_3 = 6$$
$$2x_1 - x_2 + 3x_3 = 7$$

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ -4 & 3 & -1 & 6 \\ 2 & -1 & 3 & 7 \end{array}\right] \xrightarrow[R_3+(-2)R_1]{R_2+4R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 3 & 3 & 18 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 2 & 7 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 7/2 \end{array}\right]^{\star} \xrightarrow[R_2+(-1)R_3]{R_1+(-1)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 7/2 \end{array}\right]$$

$2x_3 = 7$

This last augmented matrix is associated to the new system

$x_1 = -1/2$
$x_2 = 5/2$.
$x_3 = 7/2$

So, the solution set for the original system is just

$$\left\{ \begin{bmatrix} -1/2 \\ 5/2 \\ 7/2 \end{bmatrix} \right\}.$$

⋆ Note: I could have stopped at this point and used back-substitution to solve!

⋆ **Note.** This algorithm for solving SLEs will be one of your primary tools in this course; be sure to practice it (and understand what your computations actually mean)!