May 10 (Lecture 2)

Overview: Today we'll keep working with vectors in $\mathbb{R}^n$, in particular noting some geometric aspects of vectors.

Learning Goals:

- Correctly define and do basic operations with vectors in $\mathbb{R}^n$.
- Use the dot product to compute lengths of and angles between vectors.

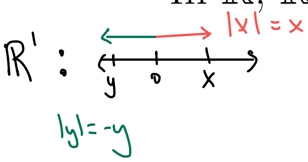
As you're getting settled:

- You should have received an e-mail for "Homework 0", which is a practice assignment for submitting to Crowdmark! Give it a shot.
- Homework 1 will be released tomorrow morning and due next Tuesday (at **11:30 pm** sharp).
- Office Hours schedule! (Also posted on Brightspace)
  - Mondays: 4:30 - 5:30pm
  - Wednesdays: 1:30 - 2:30pm
  - Fridays: 11:30-12:30pm

Length/Angles of Vectors: The Dot Product

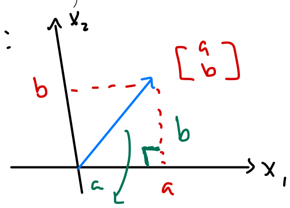
In $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, vectors have obvious "length":

$\mathbb{R}^1$:



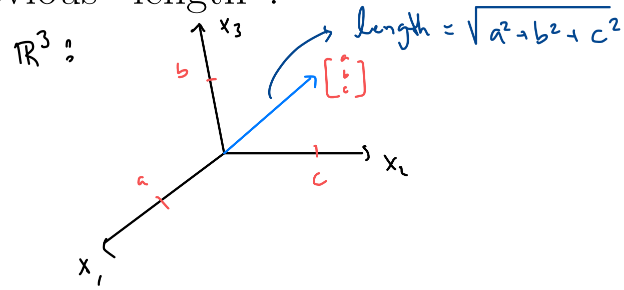
$\hookrightarrow \sqrt{y^2}$

$\mathbb{R}^2$:



$\sqrt{a^2+b^2}$

$\mathbb{R}^3$:



p. 60 **Definition.** Let $\vec{x} \in \mathbb{R}^n$. The *norm* of $\vec{x}$, denoted by $\|\vec{x}\|$, is

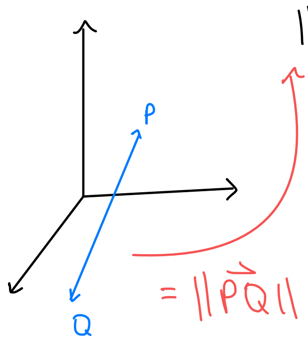
$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \boxed{\sqrt{\sum_{i=1}^{n} x_i^2}}$$

"The abs. value in $\mathbb{R}$ is the norm of a vector in $\mathbb{R}$."

Example.

- In $\mathbb{R}^4$: If $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix}$, then $\|\vec{x}\| = \sqrt{2^2 + 0^2 + 1^2 + (-3)^2} = \sqrt{4+1+9}$ $\boxed{= \sqrt{14}.}$

- Find the distance between $P(1,1,1)$ and $Q(2,0,-3)$ in $\mathbb{R}^3$.



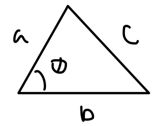
$$\|\vec{QP}\| = \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\|$$

$$= \sqrt{1+1+16}$$

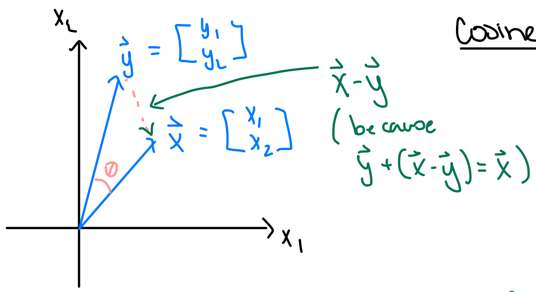
$$= \sqrt{18}$$

$$\boxed{= 3\sqrt{2}}$$

p. 61 **Definition.** A vector $\vec{x} \in \mathbb{R}^n$ is called a *unit vector* when $\|\vec{x}\| = 1$.



Question. What's the angle between two vectors in $\mathbb{R}^2$?



<u>Cosine Law:</u> $c^2 = a^2 + b^2 - 2ab\cos\theta$

$\hookrightarrow \|\vec{x} - \vec{y}\|^2 = \|\vec{y}\|^2 + \|\vec{x}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

<u>Expand:</u> $(x_1 - y_1)^2 + (x_2 + y_2)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

<u>Cancel:</u> $-2x_1y_1 - 2x_2y_2 = -2\|\vec{x}\|\|\vec{y}\|\cos\theta$

<u>Divide by -2:</u> $x_1y_1 + x_2y_2 = \|\vec{x}\|\|\vec{y}\|\cos\theta$.

★ Cosine law and the norm.

This fancy quantity looks like it could be important, so let's name it.

p.60 **Definition.** Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. The <u>*dot product*</u> (or <u>*scalar product*</u>) of $\vec{x}$ with $\vec{y}$, denoted $\vec{x} \cdot \vec{y}$, is

$$\vec{x} \cdot \vec{y} = x_1y_1 + \dots + x_ny_n = \boxed{\sum_{i=1}^{n} x_iy_i}$$

So, for vectors not just in $\mathbb{R}^2$ but in any $\mathbb{R}^n$, we have:

$$\boxed{\vec{x} \cdot \vec{y} = \|\vec{x}\| \cdot \|\vec{y}\| \cos\theta}$$

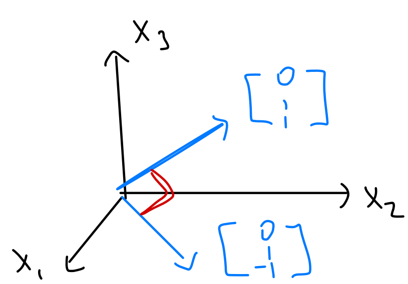
Example.

- $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = (1)(0) + (2)(1) + (-1)(2) + (3)(3) = 0 + 2 - 2 + 9 = \boxed{9}$

  Angle between: $\theta = \arccos\left(\dfrac{9}{\sqrt{15}\sqrt{14}}\right) \approx \boxed{0.9}$

- $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = (0)(0) + (1)(1) + (1)(-1) = 1 - 1 = \boxed{0}$

  Angle between: $\theta = \arccos\left(\dfrac{0}{\sqrt{2}\sqrt{2}}\right) = \boxed{\dfrac{\pi}{2}}$



• $\vec{0} \in \mathbb{R}^n$ is orthogonal to <u>every</u> vector in $\mathbb{R}^n$!

p. 62 **Definition.** Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are *orthogonal* when $\vec{x} \cdot \vec{y} = 0$

Example.

In $\mathbb{R}^5$: $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \cdot \begin{bmatrix}0\\-1\\-1\\0\\1\end{bmatrix} = (1)(0)+(2)(-1)+(3)(-1)+(4)(0)+(5)(1)$

$= -2-3+5 = 0$

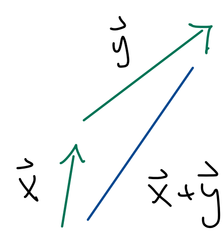
★ These two vectors are orthogonal!

Very Important!

p. 60-61

Theorem. *(1.5.1, 1.5.2) For every $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $s, t \in \mathbb{R}$, we have:*

- $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2 \geq 0$.
- ★ $\vec{x} \cdot \vec{x} = 0$ *if and only if* $\vec{x} = \vec{0}$.
- $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$. ⟵ Symmetric Property
- $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s\vec{x} \cdot \vec{y} + t\vec{x} \cdot \vec{z}$. ⟵ "Linearity"!
- ★ $\|\vec{x}\| \geq 0$ *and* $\|\vec{x}\| = 0$ *if and only if* $\vec{x} = \vec{0}$.
- $\|t\vec{x}\| = |t| \, \|\vec{x}\|$.
- $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$. "Cauchy Schwarz inequality"
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. ⟵ <u>Triangle Inequality!</u>



Example. Let $\vec{x} = \begin{bmatrix}2\\0\\-1\end{bmatrix}$, $\vec{y} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$, and suppose that $\vec{z}$ is such that $\vec{x} \cdot \vec{z} = 0$. Compute $\vec{x} \cdot (3\vec{z} - \vec{y})$.

<u>Linearity Property:</u>

$\vec{x} \cdot (3\vec{z} - \vec{y}) = 3(\underbrace{\cancel{\vec{x} \cdot \vec{z}}}_{0}) - \vec{x} \cdot \vec{y} = -\begin{bmatrix}2\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\end{bmatrix} = -(2+0+1)$

$\boxed{= -3}$

Hint: We can't divide by a vector!

Note: Need to be same $\mathbb{R}^n$, and if not it isn't defined.

norm 1

Example. Find a unit vector in the same direction as $\vec{x} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$.

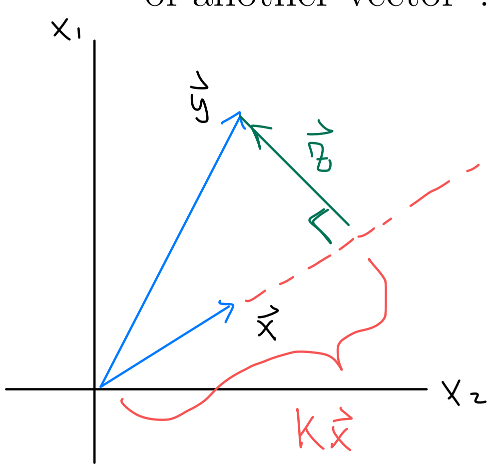
Divide $\vec{x}$ by $\|\vec{x}\| = \sqrt{1+4+9} = \sqrt{14}$: $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\-3\end{bmatrix} = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\-3/\sqrt{14}\end{bmatrix}$

Is it a unit vector? Yes!

$$\left\|\frac{1}{\|\vec{x}\|}\vec{x}\right\| = \frac{1}{\cancel{\sqrt{14}}}\cdot\cancel{\|\vec{x}\|} = 1.$$

Projections in $\mathbb{R}^n$

Often we want to find out "how much of one vector is in the direction of another vector".



Based on our picture, $\vec{y} = k\vec{x} + \vec{z}$. What is k?

Take the dot product on both sides with $\vec{x}$:

$$\vec{x}\cdot\vec{y} = \vec{x}(k\vec{x}+\vec{z}) = k\underbrace{\vec{x}\cdot\vec{x}}_{=\|\vec{x}\|^2} + \cancel{\vec{x}\cdot\vec{z}}^{=0} = k\|\vec{x}\|^2.$$

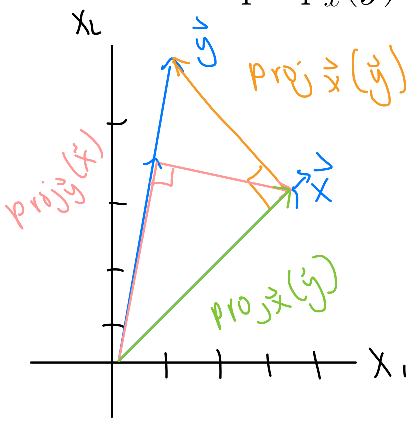
$$\Rightarrow k = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2} = \frac{\vec{x}\cdot\vec{y}}{\vec{x}\cdot\vec{x}}.$$

So, $\vec{y} = \underbrace{\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)}\vec{x} + \left(\text{some vector orthogonal to } \vec{x}\right).$

p.64 **Definition.** Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $\vec{x} \neq \vec{0}$. The *projection of* $\vec{y}$ *onto* $\vec{x}$, denoted $\text{proj}_{\vec{x}}(\vec{y})$, is the vector $\underbrace{\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)}_{\text{scalar}}\cdot\underbrace{\vec{x}}_{\text{vector}}$. (scalar multiplication)

The *projection of* $\vec{y}$ *orthogonal to* $\vec{x}$ (or *perpendicular part*), denoted $\text{perp}_{\vec{x}}(\vec{y})$, is the vector $\vec{y} - \left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\right)\vec{x}$ → dot product to zero.

Example. Let $\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Compute $\text{proj}_{\vec{x}}(\vec{y})$, $\text{proj}_{\vec{y}}(\vec{x})$, and $\text{perp}_{\vec{x}}(\vec{y})$.



$$\text{proj}_{\vec{x}}(\vec{y}) = \frac{\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix}}{16+9} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{19}{25} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\text{proj}_{\vec{y}}(\vec{x}) = \frac{\begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}}{1+25} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{19}{26} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

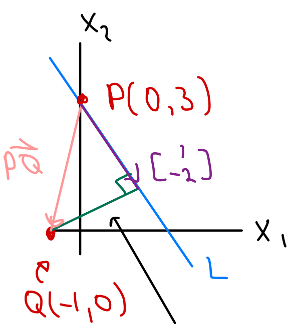
$$\text{Perp}_{\vec{x}}(\vec{y}) = \vec{y} - \text{proj}_{\vec{x}}(\vec{y}) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \frac{19}{25} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \dots$$

$$= \begin{bmatrix} -31/25 \\ 68/25 \end{bmatrix}$$

P.71 (see posted notes for a stats example)

Examples continued, and applications:

Consider the line $L: x_2 = -2x_1 + 3$. What is the minimum distance from $Q(-1,0)$ to L?



Vector EQ for L: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1+3 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\text{direction of } L!} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. $(x_1 \in \mathbb{R})$

The minimum distance will be ...

$$\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \|\vec{PQ} - \text{proj}_{\vec{d}}(\vec{PQ})\|$$

$$= \left\| \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{= \begin{bmatrix} -1 \\ -3 \end{bmatrix}} - \left(\frac{\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix}}{1+4} \right) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ -3 \end{bmatrix} - \frac{5}{5} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right\| = \sqrt{4+1}$$

$$\boxed{= \sqrt{5}}$$