May 6 (Lecture 1)

Overview: Welcome to MATH 211! Today we will give an overview of the course structure and then get right into some foundational content: vectors in $\mathbb{R}^n$!

Learning Goals:

- Be familiar and comfortable with the course and assessment structure.
- Correctly define and do basic operations with vectors in $\mathbb{R}^n$.

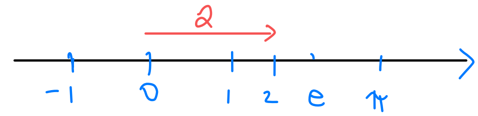
As you're getting settled:

- (Here's where I would normally put notes about course scheduling, etc.)

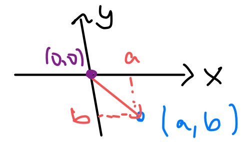
- 30 second Stretch break!
- Friday's Office Hours: 1:00pm - 2:00pm

Let's remind ourselves about what we know: real numbers!

How do we represent $\underbrace{\text{real numbers}}_{\text{Set of \#: } \mathbb{R}}$ visually?



Pairs of real #'s: on a plane! "$\mathbb{R}^2$"

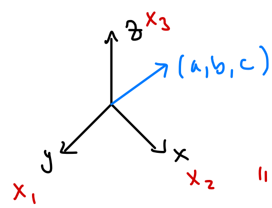


other notations: $a\hat{i} + b\hat{j}$

$\begin{bmatrix} a \\ b \end{bmatrix}$ "column vector"

Triples of real #'s: "$\mathbb{R}^3$"

"real life space"



or: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

"right-handed" coordinate system

Sometimes we need more than what we can draw, though:

- length of 200 sea cucumbers $\mathbb{R}^{200}$
- position and momentum of a particle $\mathbb{R}^6$
- space and time coords $\mathbb{R}^4$
- average temp. of one million grams of water $\mathbb{R}^{1\,000\,000}$
- Relevance of websites to your search term $\mathbb{R}^{\text{lots}}$

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Definition. The set of *vectors* in $\mathbb{R}^n$ is the collection of $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where each x_i is a real number. ($x_i \in \mathbb{R}$).

In notation $\mathbb{R}^n = \left\{ \begin{bmatrix} x_i \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \right\}$ "in"

"set" " where " or "such that"

If $\bar{x} = \begin{bmatrix} x_i \\ \vdots \\ x_n \end{bmatrix}$, x_i is the i^{th} component or entry of $\bar{x}$.

If $\bar{x}, \bar{y} \in \mathbb{R}^n$ and each $x_i = y_i$, then $\bar{x}$ and $\bar{y}$ are equal: $\bar{x} = \bar{y}$.

Example. $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ as before!

In $\mathbb{R}^5$: $\bar{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \\ 42 \end{bmatrix} \in \mathbb{R}^5$. If $\bar{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, then $\bar{x} \neq \bar{y}$. (3rd and 5th components are not equal).
($n=5$)

Vectors aren't just for "keeping track" of a collection of real numbers; their utility comes from the fact that we can combine them!

Definition. Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ and let α be a real number (or a *scalar*).

"alpha"

The sum of $\bar{x}$ and $\bar{y}$ is $\bar{x} + \bar{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$. "Vector Addition"

The scalar multiple of $\bar{x}$ by α is $\alpha \bar{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$. "Scalar multiplication"

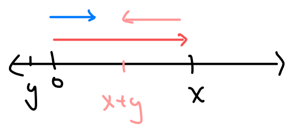
In $\mathbb{R}^2$: $\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}1+(-2)\\2+1\end{bmatrix}=\boxed{\begin{bmatrix}-1\\3\end{bmatrix}}$

Example. $5\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}5(3)\\5(1)\end{bmatrix}=\boxed{\begin{bmatrix}15\\5\end{bmatrix}}$

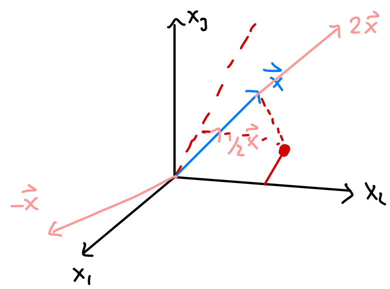
$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}-2\\1\end{bmatrix}+5\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}+\begin{bmatrix}15\\5\end{bmatrix}=\boxed{\begin{bmatrix}14\\8\end{bmatrix}}$$

Pictures. For $n = 1, 2, 3$ we can draw pictures!

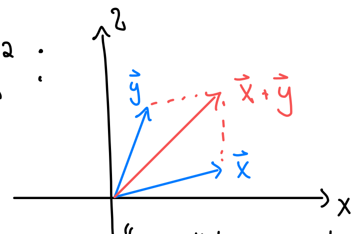
$\mathbb{R}^1$:



$\mathbb{R}^3$:



$\mathbb{R}^2$:



"Parrallelogram rule" for vector addition

p.49 **Definition.** Let $\vec{v}_1, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$ and let $c_1, \ldots, c_k$ be real numbers. The <u>*linear combination*</u> of $\vec{v}_1, \ldots, \vec{v}_k$ with scalars $c_1, \ldots, c_k$ is

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ex. $c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \ldots + c_k \cdot \vec{v}_k$

Example.

In $\mathbb{R}^2$: if $\vec{v}_1=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-2\\1\end{bmatrix}$, $\vec{v}_3=\begin{bmatrix}3\\-1\end{bmatrix}$; $c=1=c_2$, $c_3=5$,

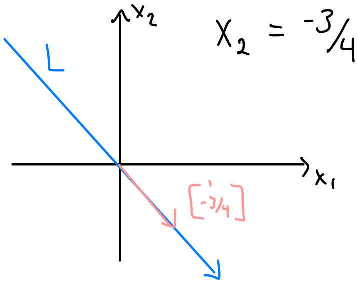
then: $\underbrace{1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-2\\1\end{bmatrix}+5\begin{bmatrix}3\\-1\end{bmatrix}}_{\text{linear combination!}}=\begin{bmatrix}-1\\3\end{bmatrix}+\begin{bmatrix}15\\-5\end{bmatrix}=\boxed{\begin{bmatrix}14\\-2\end{bmatrix}}$

In $\mathbb{R}^4$: $\vec{v}_1=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}2\\-2\\4\\0\end{bmatrix}$, $c=-1$, $c_2=\frac{1}{2}$, then: $c_1\vec{v}_1+c_2\vec{v}_2=$

$$\begin{bmatrix}-1\\-2\\-3\\-4\end{bmatrix}+\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}=\boxed{\begin{bmatrix}0\\-3\\-1\\-4\end{bmatrix}}$$

Question. Why do we call them "linear" combinations?

Answer. How do we describe this line in $\mathbb{R}^2$ using vectors?



$x_2 = -3/4\, x_1$

↳ For vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -3/4\, x_1 \end{bmatrix} = \boxed{x \begin{bmatrix} 1 \\ -3/4 \end{bmatrix}}$.

$+ \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

* wouldn't have change the line.

Vectors on L are all scalar multiples of $\begin{bmatrix} 1 \\ -3/4 \end{bmatrix}$!

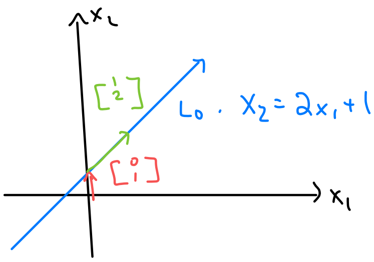
- $3/4\, x_1 + x_2 = 0$

or $3x_1 + 4x_2 = 0$

$\boxed{\circ \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} \text{ * "in terms of } x_1\text{"}}$

$$L = \left\{ x_1 \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} : x_1 \in \mathbb{R} \right\}.$$

- What about this line?

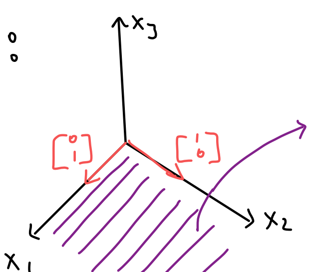


$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{= x_1 \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{direction vector}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{a point/vector on the line.}}}$$

"Vector equation for L_0".

In $\mathbb{R}^3$:



$$\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \boxed{s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

* $x_1 x_2$-plane is all linear combinations of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$!

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Theorem (1.4.1)**.** *For all $\vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, we have:*

1. $\vec{v} + \vec{w} \in \mathbb{R}^n$
2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. $(\vec{v} + \vec{w}) + \vec{x} = \vec{v} + (\vec{w} + \vec{x})$
4. *There is a vector* $\vec{0}$ *(the* zero *vector) such that* $\vec{v} + \vec{0} = \vec{v}$.
5. "additive inverse" → *For each* $\vec{v}$ *there is* $-\vec{v} \in \mathbb{R}^n$ *such that* $\vec{v} + (-\vec{v}) = 0$
6. $t\vec{v} \in \mathbb{R}^n$
7. $s(t\vec{v}) = (st)\vec{v}$
8. $(s + t)\vec{v} = s\vec{v} + t\vec{v}$
9. $t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$
10. $1\vec{v} = \vec{v}$.

"distributive properties" (8, 9)

★ These rules help you do computations not just "concretely" but also "abstractly"!

Notes.

- Zero Vector: $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (easy to check) (Not empty!)
- What should $-\vec{v}$ be? If $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, then $-\vec{v} = \begin{bmatrix} -v_1 \\ \vdots \\ -v_n \end{bmatrix}$.
- We can use these equations in both "directions".

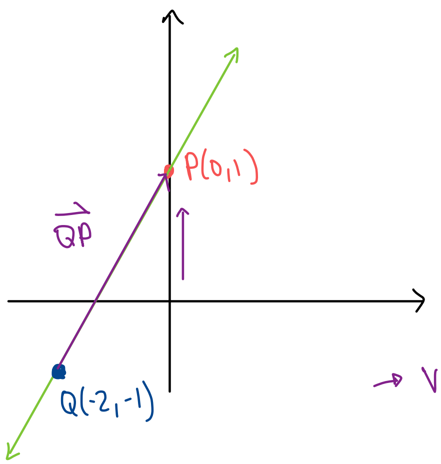
$$\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

[solutions added in "post-production"]
c.f. pg. 8-14 in the textbook

More examples.

"that goes through"

• Let L be the line between P(0,1) and Q(-2,-1) in $\mathbb{R}^2$.
Find the vector equation of L. Is R(3,4) on L?



• What is the direction of the line?
↳ Try finding $\vec{QP}$.

$$\vec{QP} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \leftarrow \text{"direction vector" for } L.$$

Now, translate by a point on the line, say p: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

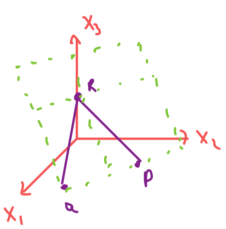
→ Vector EQ for L: $\boxed{\vec{x} = t\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

point on the line
some scalar vector
direction

Is R(3,4) on L?
i.e. $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = t\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$\rightarrow \begin{cases} 3 = 2t \\ 4 = 2t+1 \end{cases}$ If $t = 3/2$, then yes! solves the EQ.

• Find the vector EQ of the plane containing P(1,2,0), Q(2,1,0), R(0,0,1). Is O(0,0,0) on the plane?



(see the plane totally shape?)

$$\vec{RP} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{RQ} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Vectors between points in the plane are linear combinations of $\vec{RP}$ and $\vec{RQ}$!
Then, translate by a point in the plane, e.g. R(0,0,1)! So the vector EQ for this plane is

$$\boxed{\vec{x} = s\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.}$$

Is O(0,0,0) in the plane? Well if $\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then:

$$\begin{cases} 0 = s + 2t \\ 0 = 2s + t \\ 0 = 1 - s - t \end{cases}.$$

Subtract the first EQ from the second:
$s - t = 0$ or $t = s$.

But then $t = s = 0$, so $1 = 0$... which doesn't work!
$\Rightarrow$ $\boxed{\text{No}}$, not in the plane!